RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited. Manuscripts more than eight typewritten double spaced pages long will not be considered as acceptable.

THE INVERSION ENUMERATOR FOR LABELED TREES

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1. One of us (C.L.M.), examining the cumulants of the lognormal probability distribution, noticed that they involve certain polynomials $J_n(x)$ of degree $\frac{1}{2}n(n-1)$, which suggests inversions (the number of inversions of a permutation is the number of transpositions needed to restore the standard order), and with $J_n(1)=n^{n-2}$, which suggests labeled trees. And indeed $J_n(x)$ is the enumerator of trees with n labeled points by number of inversions, when inversions are counted in the following way. First, the point labeled 1 is taken as a root. Then inversions are counted on each branch, ordered away from the root; the number of inversions contributed by a point labeled i on a branch or subbranch is the number of points more remote from the root with labels less than i. It will be shown that

$$J_{n+1}(x) = Y_n(K_1(x), \cdots, K_n(x)) \tag{1}$$

with $K_i(x)=(1+x+ \cdots +x^{i-1})J_i(x)$, Y_n the (E.T.) Bell multivariable polynomial, and that

$$\exp \sum_{n=1} \frac{y^n}{n!} (x-1)^{n-1}J_n(x) = \sum_{n=0} \frac{y^n}{n!} x^{C_{n,2}}. \tag{2}$$

To see the connection with the lognormal distribution, suppose ξ is a normal random variable with mean μ, variance σ^2. Then $\eta=\exp \xi$ is lognormal with $E(\eta^k)=\exp(k\mu+ \frac{1}{2}k^2\sigma^2)$, so that the cumulant generating function for η is

$$\log E(\exp t\eta) = \log \sum_{k=0} \frac{t^k}{k!} \exp(k\mu + \tfrac{1}{2}k^2\sigma^2) = \log \sum_{k=0} \frac{y^k}{k!} x^{C_{k,2}}$$

where $y=t \exp(\mu+ \frac{1}{2}\sigma^2)$, $x=\exp \sigma^2$. On expanding the right hand side in powers of y, one finds that the coefficient of y^n has a factor $(x-1)^{n-1}$, so attention becomes focused on the polynomials $J_n(x)$ appearing in (2).