ON THE ENUMERATION OF PLANAR MAPS

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A *planar map* is determined by a finite connected nonnull graph embedded in the 2-sphere or closed plane. It is permissible for the graph to have loops or multiple joins. It separates the remainder of the surface into a finite number of simply-connected regions called the *faces* of the map. We refer to the vertices and edges of the graph as the *vertices* and *edges* of the map, respectively. The *valency* of a vertex is the number of incident edges, loops being counted twice.

A *vertex-map* is a planar map having exactly one vertex and no edges. Clearly a vertex-map has only one face. A map with exactly one edge is called a *link-map* or a *loop-map* according as the two ends of the edge are distinct or coincident. Thus a link-map has exactly one face and a loop-map exactly two.

Two planar maps are *combinatorially equivalent* if there is a homeomorphism of the surface which transforms one into the other. To within a combinatorial equivalence there is only one vertex-map, one link-map and one loop-map. But the vertex-map, link-map and loop-map are combinatorially distinct from one another.

Consider a planar map M which is not a vertex-map. Each face of M has an associated *bounding path* in the graph. We can consider this to be the path traced out by a point moving along the edges of the graph in accordance with the following rules. Normally in any small interval of time the point traces out a simple arc. On one side of this directed arc, let us say the right side, there is locally nothing but points of the face. Having started along one edge, the point continues along it, without reversing direction, until it comes to the far end. If this end is monovalent, the point then proceeds back along the same edge. This behaviour at a monovalent vertex constitutes the only exception to the rule of simple arcs in short intervals of time.

The bounding path is the cyclic sequence of positions of the moving point, some of which may be repeated. We restrict it to a single cycle by the rule that it may not traverse any edge twice in the same direction. When the distinction between right and left has been made for a map, the above rules determine the bounding path of each face

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