A COMBINATORIAL COINCIDENCE PROBLEM

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Let $A \subset E^m$ $(m \geqq 1)$, let $B(o) \subset E^m$ be convex with center of symmetry at o, let n and p be integers $(1 \leqq p \leqq n,\ n \geqq 2)$, and let $f(u)$ be an integrable function defined on A. Let A^n be the Cartesian product of A with itself n times and define $Y \subset A^n$ by

$$Y = \left\{ x = (x_1, \cdots, x_n) : \bigcap_{k=1}^{p} B(x_{i_k}) \neq \varnothing \right.$$
$$\left. \text{for some } i_1, \cdots, i_p,\ 1 \leqq i_1 < \cdots < i_p \leqq n \right\}.$$

The problem of evaluating $J = \int_Y \prod_1^n f(x_i) dx_1 \cdots dx_n$ generalizes a number of questions in probability, queuing theory, scattering, statistical mechanics etc., [1], [2]. Put

$$M = \binom{n}{p},\ S_{i_1 \cdots i_p} = \left\{ (x_1, \cdots, x_n) : \bigcap_{s=1}^{p} B(x_{i_s}) \neq \varnothing \right\}, F(x)$$
$$= \prod_1^n f(x_i),\ dV = dx_1 \cdots dx_n$$

and let the M sets $S_{i_1 \cdots i_p}$ be enumerated as $\{S_k\}$, $k = 1, \cdots, M$. Then by the inclusion-exclusion principle [2]

$$(1) \qquad J = \sum_{r=1}^{n} (-1)^{r+1} \left[\sum_{1 \leqq k_1 < \cdots < k_r \leqq M} \cdots \sum \int_{S_{k_1} \cap \cdots \cap S_{k_r}} F(x) dV \right]$$
$$= \sum_{r=1}^{n} (-1)^{r+1} U_r,$$

say. To help us keep track of different r-tuples of p-tuples, we introduce a generalization of graphs. Let X be a regular simplex in E^{n-1} with the vertices $w_1, \cdots, w_n$, a (d-dimensional) hypergraph G on X is just a collection of some of the (C_{d+1}^n) d-dimensional faces of X; the number of vertices of X lying in G will be denoted by $v(G)$. G is called a (B, r)-hypergraph on X if it consists of r such d-faces and if there are some $v = v(G)$ translates $B_1, \cdots, B_v$ of B such that any $d+1$ of them, say $B_1, \cdots, B_{d+1}$, intersect if the corresponding vertices $w_1, \cdots, w_{d+1}$ lie in a d-face of X included in G.

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