A PROBLEM OF SCHMIDT AND SPITZER

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Introduction. In §1 we formulate a question posed in [5] concerning the eigenvalues of the sections of a Toeplitz form, and summarize needed results on this subject from [4] and [5]. In §2 we prove a theorem which we then apply to answer the question affirmatively.

1.1. *Toeplitz forms.* Let $\{a_n\}$, $n = \cdots, -1, 0, 1, \cdots$ be the Fourier coefficients of a bounded, measurable, complex valued function defined on $[0, 2\pi]$. The Toeplitz form $[a_{m-n}]$, $m, n = 0, 1, \cdots$, can be considered as an operator on $\{[x_n]\}$, $n = 0, 1, \cdots$, where $\sum |x_i|^2 < \infty$. Widom [6] has shown that σ^+, the spectrum of this operator, is connected, answering a question posed in [3].

1.2. *Sections of a Toeplitz form.* Let M_n denote the matrix $[a_{m-n}]$, $m, n = 0, 1, \cdots, n$. The eigenvalues of M_n, namely the zeros of $D_n(\lambda) = \det[M_n - \lambda I_n]$, are $\lambda_{0,n}, \cdots, \lambda_{n,n}$ and we denote the set by σ_n. Let $B = \{\lambda \mid \lambda = \lim \lambda_n, \lambda_n \in \sigma_{i_n}, i_n \to \infty\}$. In the Hermitian case, that is when $a_m = \bar{a}_{-m}$, there is an extensive theory of the distribution of the $\{\lambda_{i,n}\}$ [2], and in particular $B = \sigma^+$. In the non-Hermitian case the main results are found in [4] and [5] where the added hypothesis $a_n = 0$ for $|n| \geqq p > 0$ is made. We call this the doubly restricted case of Toeplitz sections, since progress has been made in the singly restricted case [1], that is when $a_n = 0$ for $n \geqq p > 0$. This may prove to be an avenue of access to the unrestricted case.

1.3. *A question on doubly-restricted Toeplitz sections.* Let $f(z) = \sum_{-k}^{h} a_\nu z^\nu$, where $a_{-k} \neq 0$ and $-k \leqq -1$. Let $\alpha_i(\lambda)$, $i = 1, \cdots, k+h$, be the moduli of the zeros of $z^k(f(z) - \lambda)$, with repetitions for multiple zeros and indexed so that $\alpha_i(\lambda) \leqq \alpha_{i+1}(\lambda)$, $i = 1, \cdots, h+k-1$. If $C = \{\lambda \mid \alpha_k(\lambda) = \alpha_{k+1}(\lambda)\}$, then it is shown in [5] that $C = B$. The question is raised whether C is connected. This will be answered in the affirmative. We note the parallel between this question and the one mentioned in 1.1.

1.4. *Some needed results.* In simplifying and extending the results of [5], Hirschman has made explicit the structure of C, which is implicit in [5].

LEMMA 1. *C can be represented as a finite union of closed analytic arcs, where either distinct arcs are disjoint, or, if not, their intersection consists of one or both common end points. None of the arcs are degenerate, i.e., points.*