DIFFERENTIABLE DYNAMICAL SYSTEMS[1]

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PART I. DIFFEOMORPHISMS

I.1. **Introduction to conjugacy problems for diffeomorphisms.** This is a survey article on the area of global analysis defined by differentiable dynamical systems or equivalently the action (differentiable) of a Lie group G on a manifold M. An action is a homomorphism $G \to \mathrm{Diff}(M)$ such that the induced map $G \times M \to M$ is differentiable. Here $\mathrm{Diff}(M)$ is the group of all diffeomorphisms of M and a diffeomorphism is a differentiable map with a differentiable inverse. Everything will be discussed here from the C^∞ or C^r point of view. All manifolds maps, etc. will be differentiable (C^r, $1 \leqq r \leqq \infty$) unless stated otherwise.

In the beginning we will be restricted to the discrete case, $G = Z$. Here Z denotes the integers, Z^+ the positive integers. By taking a generator $f \in \mathrm{Diff}(M)$, this amounts to studying diffeomorphisms on a manifold from the point of view of orbit structure. The orbit of $x \in M$, relative to f, is the subset $\{f^m(x) \mid m \in Z\}$ of M or else the map $Z \to M$ which sends m into $f^m(x)$. The finite orbits are called *periodic orbits* and their points, *periodic points*. Thus $x \in M$ is a periodic point if $f^m(x) = x$ for some $m \in Z^+$. Here m is called a period of x and if $m = 1$, x is a *fixed point*. Our problem is to study the global orbit structure, i.e., all of the orbits on M.

The main motivation for this problem comes from ordinary differential equations, which essentially corresponds to $G = R$, R the reals acting on M. There are two reasons for this leading to the diffeomorphism problem. One is that certain differential equations have cross-sections (see, e.g., [**114**]) and in this case the qualitative study of the differential equation reduces to the study of an associated diffeomorphism of the cross-section. This is the reason why Poincaré [**90**] and Birkhoff [**19**] studied diffeomorphisms of surfaces.

I believe there is a second and more important reason for studying the diffeomorphism problem (besides its great natural beauty). That is, the same phenomena and problems of the qualitative theory of ordinary differential equations are present in their simplest form in the diffeomorphism problem. Having first found theorems in the dif-

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