EXTENSION OF VALUATION THEORY

BY MERLE E. MANIS

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By a valuation on a commutative ring R with 1 we mean a pair (v, Γ) where Γ is an ordered (multiplicative) group with zero adjoined and v is a map from R *onto* Γ satisfying

(1) $v(xy)=v(x)v(y)$ for all $x, y \in R$,

(2) $v(x+y) \leqq \max \{v(x), v(y)\}$ for all $x, y \in R$.

This generalizes the field concept; the insistence on "onto" is what allows us to generalize the main field theorems.

PROPOSITION 1. *Let A be a subring of a ring R, P a prime ideal of A. Then the following are equivalent:*

(1) *For each subring B of R and prime ideal Q of B with $A \subset B$, $Q \cap A = P$, one has $A = B$.*

(2) *For $x \in R \setminus A$ there exists a $y \in P$ with $xy \in A \setminus P$.*

(3) *There is a valuation (v, Γ) on R with*

$$A = \{x \in R \mid v(x) \leqq 1\}, \qquad P = \{x \in R \mid v(x) < 1\}.$$

We call pairs (A, P) satisfying the three equivalent conditions *valuation pairs.*

PROPOSITION 2. *The valuations (v, Γ) and (w, Λ) determine the same valuation pair (A, P) if and only if there is an order isomorphism ϕ of Γ onto Λ such that $w = \phi \circ v$.*

Let the valuation (v, Γ) determine the valuation pair (A, P). Then an ideal $\mathfrak{A}$ of A is called *v-closed* if $x \in \mathfrak{A}$, $y \in R$ and $v(y) \leqq v(x)$ implies $y \in \mathfrak{A}$.

PROPOSITION 3. *The v-closed ideals of A are linearly ordered by inclusion. The v-closed prime ideals are in* 1-1 *correspondence with the isolated subgroups of Γ. If $\phi: \Gamma \rightarrow \Gamma/\Sigma$ is the natural map with Σ an isolated subgroup of Γ, then the v-closed prime ideal corresponding to Σ is the ideal of the valuation pair determined by the valuation $(\phi \circ v, \Gamma/\Sigma)$.*

Independence and dominance of valuations are defined as in [5] and the "same" computational lemmas are obtained.

Let R be a ring extension of a ring K, (v_0, Γ_0) a valuation on K. By an extension of (v_0, Γ_0) to R we mean a valuation (v, Γ) on R and an order isomorphism ϕ of Γ_0 into Γ such that $v(x) = \phi \circ v_0(x)$ for all $x \in K$.