IS EVERY INTEGRAL NORMAL?

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1. **Introduction.** For the terminology and notation not explained below the reader is referred to [1] and [2].

Let L be a Riesz space. A positive linear functional $0 \leqq \phi \in L^{\sim}$ is called an *integral* whenever for every decreasing sequence $\{u_n\}$ of nonnegative elements of L, $\inf_n u_n = 0$ implies $\inf_n \phi(u_n) = 0$. A positive linear functional $0 \leqq \phi \in L^{\sim}$ is called *normal* whenever $0 \leqq u_\tau \in L$ and $u_\tau \downarrow 0$ (i.e., for every pair u_{τ_1}, u_{τ_2} there is an element u_{τ_3} such that $u_{\tau_3} \leqq \inf(u_{\tau_1}, u_{\tau_2})$ and $\inf u_\tau = 0$) implies $\inf_\tau \phi(u_\tau) = 0$. Finally a positive integral is called *singular* whenever $0 \leqq \chi \leqq \phi$ and χ is a normal positive linear functional implies $\chi = 0$. Every positive linear functional can be written uniquely as the sum of a normal integral and a singular integral.

From the point of view of the theory of positive linear functionals it seems natural to ask the question. Is every integral normal? The following example will show that the answer to this question is in general negative.

Let L be the Riesz space of all real bounded Borel measurable functions on the unit interval $0 \leqq x \leqq 1$ and let $\phi(f) = \int_0^1 f(x)dx$ be the positive linear functional determined by the Lebesgue integral. Then ϕ is an integral in the sense defined above but ϕ is not normal. In fact ϕ is a singular integral. Furthermore, observe that L is Dedekind σ-complete (i.e., every countable subset which is bounded above has a least upper bound).

The answer to this question turns out to be quite different if we restrict the class of Riesz spaces to be considered to a special class of Riesz spaces namely the class of Dedekind complete Riesz spaces. (A Riesz space L is called Dedekind complete whenever every nonempty subset of L which is bounded above has a least upper bound.) For this class of Riesz spaces we will indicate below that the statement "every integral is normal" is logically equivalent to the statement "every cardinal is nonmeasurable." A cardinal α is called measurable whenever there exists a probability measure on the algebra of all subsets of α such that every countable subset has measure

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