SEMIHEREDITARY RINGS

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1. **Introduction.** A ring is (right) semihereditary if every finitely-generated right ideal is projective. Chase [2] gave the first example of a ring which was right but not left semihereditary. In [7] the author constructed an example of a ring which was even right hereditary (all right ideals are projective) yet not left semihereditary.

In the other direction, P. M. Cohn, [3] and [4], has found certain classes of rings for which right semihereditary implies left semihereditary. In particular, total matrix rings over principal right ideal domains are both right and left semihereditary. In this note, among other things, we shall show that if a ring is right Noetherian and right hereditary then it is also left semihereditary.

2. **Notation and definitions.** Ring means ring with identity element, and all modules are unitary. R_n will denote the ring of all $n\times n$ matrices over the ring R. If S is a subset of the ring R, then $r(S)$ ($l(S)$) will denote the right (left) annihilator of S.

3. **Principal results.** The following sublemma is well known, and we omit the proof.

SUBLEMMA. *The following statements are equivalent:*

(1) *The ring R has no infinite set of orthogonal idempotents.*

(2) *The right (left) ideals of the form eR (Re) where e is an idempotent satisfy the ascending and descending chain conditions.*

We recall that the right ideal aR is projective if and only if $r(a)=eR$ where e is an idempotent.

THEOREM 1. *Let R be a ring in which every principal right ideal is projective and in which there is no infinite set of orthogonal idempotents. Then every right and every left annihilator is generated by an idempotent. In particular, every principal left ideal is projective.*

PROOF. Suppose $(0)\neq T=r(S)$. If $s\in S$, then $r(s)\supseteq T$. Thus, $T\subseteq hR$ where h is an idempotent. Now let L be an arbitrary (nonzero) left annihilator. $r(L)\subseteq gR$ where $g^2=g$. But then $L=l(r(L))\supseteq l(gR)$ $=R(1-g)$. Hence, any left annihilator, L, contains a nontrivial idempotent.