EMBEDDING PROJECTIVE SPACES[1]

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1. Haefliger reduced the question of embedding manifolds in the Euclidian space R^m to a homotopy problem in [6]. Since then it has been of some interest to find examples of n-manifolds which embed in R^{2n-k} for a given k. In particular great effort has been spent studying embeddings of the various projective spaces. However, the k that were thus obtained were in no cases larger than 5 or 6 (see for example [7], [8], [9]). Our purpose in this note is to indicate the proofs of the theorems that follow.

THEOREM 1. *Let* $n \equiv 7(8)$; *then* RP^n (real n-dimensional projective space) *embeds in* R^{2n-k} *where* $k \geqq 2\ [\log_2(\alpha(n))]-1$. (Here $\alpha(n)$ is the number of ones in the dyadic expansion of n.)

THEOREM 2. *If n is odd and $\alpha(n)$ is greater than* $4+2^i$, *then* CP^n (complex projective space) *embeds in* R^{4n-k} *with* $k \geqq 3+i$.

THEOREM 3. *If* $\alpha(n) \geqq 11+2^i$ *then* QP^n (quaternionic projective space) *embeds in* R^{8n-k} *where* $k \geqq 5+i$.

The detailed proof of Theorem 1 appears in [5] so in the sequel we will concentrate on giving those modifications which must be made in [5] so as to prove Theorems 2 and 3.

2. **A key lemma.** Let M^n immerse in R^{2n-r} and set $k(n)=8s+2^t-1$ (where $n+1=(2^{4s+t})c$ with c odd and $0 \leqq t \leqq 3$). Then for $n \geqq 3$ we have:

LEMMA 2.1. (a) *If n is odd there are exactly two isotopy classes of immersions $M^n \subseteq R^{2n}$. One contains an embedding and the other an immersion with a single double point as its only singularity, but both normal bundles have k independent cross-sections where* $k=\min(r, k(n))$.

(b) *If n is even and M^n orientable then there are Z isotopy classes of immersions $M^n \subseteq R^{2n}$ only one of which contains an embedding. The only immersion with a normal field is the embedding, hence the embedding has r normal fields.*

REMARK. Part b is false for nonorientable manifolds for all n [4].

PROOF. Part a follows from Whitney's well known results [10] on embeddings and immersions in R^{2n}, and a careful study of how one

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