PIECEWISE MONOTONE POLYNOMIAL INTERPOLATION

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The purpose of this paper is to prove the following:

THEOREM. *Suppose that n is a positive integer, $x_{i-1}<x_i$ and $y_{i-1}\neq y_i$, $i=1, 2, \cdots, n$. Then there exists a polynomial P such that $P(x_i)=y_i$, $i=0, 1, \cdots, n$ and P is monotone in each of the intervals $[x_{i-1}, x_i]$, $i=1, 2, \cdots, n$.*

This theorem differs from the usual polynomial interpolation theorem in that there is no mention of the degree of the polynomial. The proof presented here leaves little room for generalization from ordinary polynomials to more general systems of functions since it is essential that zeros can be prescribed.

PROOF. Let $y_0=0$ and generality is not lost. $D=\{Q$: Q is a polynomial and $Q(x)(y_i-y_{i-1})\geqq 0$ for $x_{i-1}\leqq x\leqq x_i$, $i=1, 2, \cdots, n\}$. If $Q_1,Q_2\in D$ and $0\leqq t\leqq 1$, then $[tQ_1(x)+(1-t)Q_2(x)]\ (y_i-y_{i-1}) =tQ_1(x)(y_i-y_{i-1})+(1-t)Q_2(x)(y_i-y_{i-1})\geqq 0$ for $x_{i-1}\leqq x\leqq x_i$ and $i=1, 2, \cdots, n$. Therefore D is a convex subset of the space $C[x_0, x_n]$. Furthermore, if $Q\in D$ and $a\geqq 0$, then $aQ\in D$.

Let F be the function from D into E_n defined by

$$F(Q) = \left\{\int_{x_0}^{x_i} Q(x)dx\right\}_{i=1}^{n}.$$

F is linear and so $F(D)$ is a convex subset of E_n. Furthermore if $z\in F(D)$ and $a\geqq 0$, then $az\in F(D)$.

For each i, $1\leqq i\leqq n$, let ϕ_i denote the point of E_n such that if $1\leqq j<i$, then the jth coordinate is zero and if $i\leqq j\leqq n$, then the jth coordinate is one. Let $\lambda_i=\text{sign}\ (y_i-y_{i-1})\cdot\phi_i$ and note that $\phi_i=\text{sign}\ (y_i-y_{i-1})\cdot\lambda_i$. Any point $Z=(z_1, z_2, \cdots, z_n)$ has the representation $z=\sum (z_i-z_{i-1})\phi_i$ where $z_0=0$. In particular, if $Q\in D$,

$$F(Q) = \sum\left(\int_{x_{i-1}}^{x_i} Q(x)dx\right)\cdot\phi_i = \sum\left(\int_{x_{i-1}}^{x_i} Q(x)dx\right)\text{sign}(y_i - y_{i-1})\cdot\lambda_i.$$

But

$$\text{sign}\left(\int_{x_{i-1}}^{x_i} Q(x)dx\right) = \text{sign}(y_i - y_{i-1})$$