SPLINE FUNCTION APPROXIMATIONS FOR SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction and description of method. The known discrete variable methods for the solutions of differential equations (see [**1**]) furnish the approximate solutions as discrete tabular values at usually equidistant values of the independent variable. The object of this paper is to search for approximate solutions in the form of a spline function $S(x)$, [**4**], of degree m $(m \geqq 2)$ and class C^{m-1}. This approach was suggested to the authors by I. J. Schoenberg [**3**].

Let the differential equation be

$$y' = f(x, y), \qquad 0 \leqq x \leqq b, \tag{1}$$

about which we assume the following. If $T = \{(x, y) \mid 0 \leqq x \leqq b\}$, then we assume that $f(x, y) \in C^{m-2}$ in T and that it satisfies the Lipschitz condition

$$|f(x, y) - f(x, y^*)| \leqq L|y - y^*| \qquad \text{if } 0 \leqq x \leqq b. \tag{2}$$

If $m \geqq 3$ then (2) is equivalent to the boundedness of $\partial f/\partial y$ in T.

Our construction of the approximate solution $S(x) = S_m(x)$ is as follows. Let $y(x)$ be the solution of (1) determined by the initial value $y(0) = y_0$. Let n be an integer $> m$, $h = b/n$ and let $S(x)$ $(0 \leqq x \leqq b)$ be a spline function of degree m, class C^{m-1} and having its knots at the points $x = h, 2h, \cdots, (n-1)h$.

We define the first component of $S(x) = S_m(x)$ by

$$\begin{aligned} S(x) = y(0) + y'(0)x + \cdots + \frac{1}{(m-1)!} y^{(m-1)}(0)x^{m-1} \\ + \frac{1}{m!} a_0 x^m \qquad (0 \leqq x \leqq h) \end{aligned} \tag{3}$$

with the last coefficient a_0 as yet undetermined. We now determine a_0 by requiring that $S(x)$ should satisfy (1) for $x = h$. This gives the equation

$$S'(h) = f(h, S(h)) \tag{4}$$

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