ON LOCAL TIME FOR MARKOV CHAINS

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The object of this paper is to present one of a class of formulae which express the time spent in an instantaneous state of a Markov chain in terms of the time spent in "neighbouring" states. Though these formulae do lead to new analytic results—some elementary consequences are given in [3]—their main use is in the analysis of sample function behaviour in the neighbourhood of an instantaneous state. Time is *always* shared out "properly" among states in such a neighbourhood. Though the particular theorem stated below refers to a real state b, it has a valid extension to the case when b is an instantaneous *fictitious* state in the sense of Neveu [2]. A detailed account of these topics will appear elsewhere.

The terminology and notation used here are exactly as in Chung [1]; see particularly the appendix for the definition of the functions g and G.

Suppose that $\{x(t): t \geqq 0\}$ is a Borel measurable, well-separable M.C. with minimal state-space I. For any state k in I, define

$$\beta_k(t, \omega) = \mu\{u: 0 \leqq u \leqq t, x(u, \omega) = k\},$$

μ denoting Lebesgue measure.

THEOREM. *Suppose that b is an instantaneous state of $\{x(t)\}$. Suppose also that $\{H_n\}$ is a sequence of subsets of $I - \{b\}$ and that $\{s_n\}$ is a sequence of positive real numbers such that, as $n \to \infty$,*

$$s_n \downarrow 0, \qquad \sum_{j \in H_n} g_{bj}(s_n) \to \infty.$$

Then, for every t,

$$\lim_{n \to \infty} \frac{\sum_{j \in H_n} g_{bj}(s_n) \beta_j(t, \omega) / G_{bj}(\infty)}{\sum_{j \in H_n} g_{bj}(s_n)} = \beta_b(t, \omega)$$

in probability.

Note. Sequences $\{H_n\}$ and $\{s_n\}$ with the stated properties must

[1] This work was done at the University of Durham, England.