STRUCTURE OF A CLASS OF REGULAR SEMIGROUPS AND RINGS

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One of the most natural approaches to the study of regular semigroups is to impose restrictions on the partial ordering of their idempotents ($e \leqq f \Leftrightarrow ef = fe = e$). The principal object of this note is to describe the structure of the classes of regular semigroups whose idempotents form a tree or a unitary tree, respectively (see Definition 1). We determine, among other things, a complete set of invariants, isomorphisms, the group of automorphisms, and congruences of these semigroups. We also consider regular rings whose multiplicative semigroup satisfies conditions (C) or (C_1) and give their structure. The terminology concerning semigroups is that of [2] and concerning p.o. sets of [1]. We consider only semigroups with zero; the statements concerning semigroups without zero can then be easily deduced.

DEFINITION 1. A [*unitary*] *tree* T is a p.o. set with a unique minimal element 0 [and a unique maximal element 1], $T \neq \{0\}$, satisfying

(i) all elements [different from 1] are of finite height;

(ii) every element different from 0 [and different from 1] covers exactly one element.

DEFINITION 2. A regular semigroup whose p.o. set of idempotents is a tree [unitary tree] is said to be $\mathfrak{I}$-*regular* [$\mathfrak{I}_1$-*regular*].

A $\mathfrak{I}_1$-regular semigroup has an identity element. In order to find the structure of such semigroups, we need the following construction. For any semigroup S with zero, we write $S^* = S \backslash 0$.

Let T be a tree; to every $\alpha \in T^* = T \backslash 0$ associate a semigroup S_α with zero 0_α; the semigroups S_α are pairwise disjoint. If $\alpha\chi > 1$ ($\alpha\chi$ is the height of α in T), associate to α a partial homomorphism ϕ_α: $S_\alpha^* \rightarrow S_{\bar{\alpha}}^*$ ($\bar{\alpha}$ is the unique element of T covered by α). On the set $V = (\bigcup_{\alpha \in T^*} S_\alpha^*) \cup 0$, multiplication is defined by induction on the height of $\alpha \in T$ as follows. Let 0 act as the zero of V. If $\alpha\chi = \beta\chi = 1$ and $x \in S_\alpha^*$, $y \in S_\beta^*$ (multiplication in S_α is denoted by juxtaposition), let

$$\begin{aligned} x \circ y &= xy && \text{if } \alpha = \beta,\ xy \neq 0_\alpha, \\ &= 0 && \text{if } \alpha = \beta,\ xy = 0_\alpha \text{ or } \alpha \neq \beta. \end{aligned}$$

Supposing that multiplication has been defined for all $u \in S_\gamma^*$, $v \in S_\delta^*$, $\gamma\chi$, $\delta\chi < n$ ($n > 1$), for $x \in S_\alpha^*$, $y \in S_\beta^*$ with $\alpha\chi$, $\beta\chi \leqq n$, let