A COMMUTATIVE SEMISIMPLE ANNIHILATOR BANACH ALGEBRA WHICH IS NOT DUAL

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Barnes [1] has constructed an example of a commutative semisimple normed annihilator algebra which is not a dual algebra. His example is not complete and when completed acquires a nonzero radical. In this paper we construct an example which is complete. The theory of annihilator algebras is developed e.g. in [2].

We put $\alpha_i = (1+(1+i)^{1/2})^{-2}$ for $i \geqq 1$ and denote by A_0 the algebra of doubly infinite sequences a with $a_i = 0$ for all but a finite number of values of i, with coordinatewise addition and multiplication. We define a norm on A_0 by

$$\|a\| = 3\left(\sum_{n\leqq 0} |a_n|^2\right)^{1/2} + 3 \sup_{n>0} \left| a_n\alpha_n^{-1} - \sum_{j=-n}^{0} a_j \right|.$$

This is easily seen to be a linear space norm on A_0 and we have that

(i)
$$\left(\sum_{n\leqq 0} |a_n b_n|^2\right)^{1/2} \leqq \left(\sum_{n\leqq 0} |a_n|^2\right)^{1/2}\left(\sum_{n\leqq 0} |b_n|^2\right)^{1/2} \leqq \tfrac{1}{9}\|a\|\,\|b\|;$$

(ii) if $n > 0$,

$$\tfrac{1}{3}\|a\| \geqq \left| a_n\alpha_n^{-1} - \sum_{j=-n}^{0} a_j \right| \geqq |a_n|\,\alpha_n^{-1} - (n+1)^{1/2}\tfrac{1}{3}\|a\|$$

so that

$$|a_n|\,\alpha_n^{-1} \leqq \tfrac{1}{3}(1+(n+1)^{1/2})\|a\|$$

and

$$|a_n b_n \alpha_n^{-1}| \leqq \tfrac{1}{9}\alpha_n(1+(n+1)^{1/2})^2\|a\|\,\|b\| = \tfrac{1}{9}\|a\|\,\|b\|;$$

(iii)
$$\left|\sum_{j=-n}^{0} a_j b_j\right| \leqq \left(\sum_{j=-n}^{0} |a_j|^2\right)^{1/2}\left(\sum_{j=-n}^{0} |b_j|^2\right)^{1/2} \leqq \tfrac{1}{9}\|a\|\,\|b\|.$$