ON THE ERGODIC THEOREM FOR POSITIVE OPERATORS[1]

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Communicated by M. Loève, December 22, 1966

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let T be a positive linear operator on $L_1(X, \mathcal{A}, \mu)$. The ratio ergodic theorem of Chacon-Ornstein (see [3], [7], [2]) assumes that $|T|_1$, the L_1 norm of T, is less than or equal to one. Here we discuss the behavior of the ratio under the weaker boundedness assumption (b_h). All sets and functions introduced below are assumed measurable. All relations are assumed to hold modulo sets of μ-measure zero. L_1^+ is the class of nonnegative *not identically vanishing* elements of L_1; similar conventions apply to other function spaces. $L_1(A)$ is the class of functions f with supp f (support of f), contained in A and $\int|f| < \infty$. $T^P g$ is the function $g+Tg+T^2g+ \cdots$. The function $f\cdot 1_A$ is sometimes written f_A. A set A is called *closed on* a set B if $f\in L_1^+(A)$ implies $1_B\cdot Tf \in L_1(A)$. A set closed on X is called *closed.*

THEOREM 1. *Let h be a fixed function in L_∞^+ and assume that T satisfies the following condition*:

$$(\mathrm{b_h}) \qquad \sup_n \int T^n f\cdot h < \infty \qquad \text{for each } f \in L_1^+.$$

Then the space X uniquely decomposes into sets Y^h and Z^h with the following properties. The set Z^h is closed and, if $f\in L_1(Z^h)$, then

$$(1) \qquad \gamma^*(f) \overset{\text{def}}{=} \lim_n\left(\sup_j n^{-1}\sum_{i=0}^{n-1}\int(T^{i+j}|f|\cdot h\right) = 0.$$

If $f\in L_1^+(Y^h)$, then $\gamma^*(f)>0$.

THEOREM 2. *Assume* $(\mathrm{b_h})$. *The set Y^h decomposes into the conservative part YC^h and the dissipative part YD^h: for each $f\in L_1^+$, $T^Pf=0$ or ∞ on YC^h; $T^Pf<\infty$ on YD^h. The subsets of YC^h closed on Y^h form a σ-field, say $\mathcal{C}^h$, and $YC^h\in\mathcal{C}^h$. If $X\neq Z^h$, then the equation*

$$(2) \qquad e \in \overset{+}{L_\infty}, \quad \operatorname{supp} e = Y^h, \quad T^*e = e$$

[1] Research supported in part by the National Science Foundation Grant GP-1458.
Complete proofs, and some examples will appear in Z. Wahrscheinlichkeitstheorie.