ON THE CLOSURE OF CERTAIN BANACH SPACES OF FUNCTIONS OF SEVERAL VARIABLES

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1. Statement of the main results. Our primary goal in this note is to establish the following proposition.

THEOREM 1. *Consider the Banach space $\mathcal{C}_n$ of continuous real-valued functions $f: E_n \to R$, E_n standing for the unit cube in n-dimensional Euclidean space. If ϕ and ψ are any fixed functions of $\mathcal{C}_n$ with connected level-sets intersecting pairwise in connected sets, then the subspace of superpositions $a \circ \phi + b \circ \psi$ is closed in $\mathcal{C}_n$ under the uniform norm.*

Mark by $\mathcal{B}_n$ the indicated space of superpositions. To prove the stated theorem, it suffices to verify

THEOREM 2. *Every function of $\mathcal{C}_n$ has a best uniform approximation in $\mathcal{B}_n$.*

Distinguish one of the fixed functions, say, ψ; denote its level sets by $l_\psi(t)$,

$$l_\psi(t) = \{p \in E_n : \psi(p) = t\};$$

designate by L_ψ the aggregate of level sets $l_\psi = l_\psi(t)$. Finally, set for each $f \in \mathcal{C}_n$

$$\omega(f \mid l_\psi) = \max_{p \in l} f(p) - \min_{p \in l_\psi} f(p),$$

$$\omega(f \mid \psi) = \max_{l_\psi \in L_\psi} \omega(f \mid l_\psi),$$

$$\mu(f) = \inf_{\mathcal{B}_n} \|f - a \circ \phi - b \circ \psi\|,$$

(the properties of the functional ω and related topics are investigated in [1] and [2]. Theorem 2 is proved by means of the five lemmas now formulated.

LEMMA 1. *For each $f \in \mathcal{C}_n$,*

$$\mu(f) = \tfrac{1}{2} \inf_{a \in \mathcal{C}} \omega(f - a \circ \phi \mid \psi),$$

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