INNER FUNCTIONS IN POLYDISCS[1]

BY WALTER RUDIN

Communicated by A. Zygmund, November 21, 1966

For $N=2, 3, 4, \cdots$ the polydisc U^N consists of all $z=(z_1, \cdots, z_N)$ in C^N (the space of N complex variables) such that $|z_j|<1$ for $j=1, \cdots, N$. The class of all bounded holomorphic functions in U^N is denoted by $H^\infty(U^N)$. If $f\in H^\infty(U^N)$ it is well known that the radial limits

$$f^*(z) = \lim_{r\to 1} f(rz) \tag{1}$$

exist for almost all z in the distinguished boundary T^N of U^N. Here $rz=(rz_1, \cdots, rz_N)$.

An *inner function in* U^N is, by definition, a function $g\in H^\infty(U^N)$ such that $|g^*(z)|=1$ for almost all $z\in T^N$.

The present note contains partial answers to questions such as the following: Is every $f\in H^\infty(U^N)$ (other than $f\equiv 0$) a product $f=gh$ where g is inner and both h and $1/h$ are holomorphic in U^N? (In this case we say that f and g *have the same zeros in* U^N.) If not, what are some sufficient conditions on f which guarantee the existence of such a factorization? If f does have the same zeros as some inner function g, does it follow that g can be chosen so that $f/g\in H^\infty(U^N)$?

A special role is played by those inner functions which (for lack of a better name) I propose to call *good*: An inner function g is good if

$$\lim_{r\to 1} \int_{T^N} \log |g(rz)| \, dm(z) = 0. \tag{2}$$

Here dm denotes the Haar measure of T^N.

To see some examples, consider these four classes of inner functions in U^N:

(A) Those which have continuous extensions to the closure of U^N.

(B) Rational inner functions.

(C) Finite or infinite (convergent) products of rational inner functions.

(D) Good inner functions.

In one variable, (A) = (B) and (C) = (D), since the good inner func-

[1] Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant No. 1160-66, and by the Wisconsin Alumni Research Foundation.