AN INEQUALITY WITH APPLICATIONS TO STATISTICAL ESTIMATION FOR PROBABILISTIC FUNCTIONS OF MARKOV PROCESSES AND TO A MODEL FOR ECOLOGY

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1. **Summary.** The object of this note is to prove the theorem below and sketch two applications, one to statistical estimation for (probabilistic) functions of Markov processes [1] and one to Blakley's model for ecology [4].

2. **Result.**

THEOREM. *Let $P(x)=P(\{x_{ij}\})$ be a polynomial with nonnegative coefficients homogeneous of degree d in its variables $\{x_{ij}\}$. Let $x=\{x_{ij}\}$ be any point of the domain D: $x_{ij}\geqq 0$, $\sum_{j=1}^{q_i} x_{ij}=1$, $i=1,\cdots,p$, $j=1,\cdots,q_i$. For $x=\{x_{ij}\}\in D$ let $\mathfrak{J}(x)=\mathfrak{J}\{x_{ij}\}$ denote the point of D whose i, j coordinate is*

$$\mathfrak{J}(x)_{ij}=\left(x_{ij}\left.\frac{\partial P}{\partial x_{ij}}\right|_{(x)}\right)\Big/\sum_{j=1}^{q_i} x_{ij}\left.\frac{\partial P}{\partial x_{ij}}\right|_{(x)}.$$

Then $P(\mathfrak{J}(x))>P(x)$ unless $\mathfrak{J}(x)=x$.

Notation. μ will denote a doubly indexed array of nonnegative integers: $\mu=\{\mu_{ij}\}$, $j=1,\cdots,q_i$, $i=1,\cdots,p$. x^μ then denotes $\prod_{i=1}^{p}\prod_{j=1}^{q_i} x_{ij}^{\mu_{ij}}$. Similarly, c_μ is an abbreviation for $c_{\{\mu_{ij}\}}$. The polynomial $P(\{x_{ij}\})$ is then written $P(x)=\sum_\mu c_\mu x^\mu$.

In our notation:

$$\mathfrak{J}(x)_{ij}=\left(\sum_\mu c_\mu \mu_{ij} x^\mu\right)\Big/\sum_{j=1}^{q_i}\sum_\mu c_\mu\mu_{ij}x^\mu. \tag{1}$$

We wish to prove

$$P(x)=\sum_\mu c_\mu x^\mu \leqq \sum_\mu c_\mu \prod_{i=1}^{p}\prod_{j=1}^{q_i}\mathfrak{J}(x)_{ij}^{\mu_{ij}}. \tag{2}$$

PROOF.

$$P(x)=\sum_\mu\left\{c_\mu\prod_{i=1}^{p}\prod_{j=1}^{q_i}\mathfrak{J}(x)_{ij}^{\mu_{ij}}\right\}^{1/d+1}$$
$$\times\left\{c_\mu^{d/d+1}\, x^\mu\prod_{i=1}^{p}\prod_{j=1}^{q_i}\left(\frac{1}{\mathfrak{J}(x)_{ij}}\right)^{\mu_{ij}/d+1}\right\}.$$