AN EXAMPLE IN THE CALCULUS OF FOURIER TRANSFORMS

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0. The functions which operate on Fourier or Fourier-Stieltjes transforms have been investigated by Helson, Kahane, Katznelson, and Rudin, especially in [1]. In this note we give an example of a positive measure on the Cantor group D_2, whose Fourier-Stieltjes transform has range in $[0, 1]$, and on which the continuous functions operating must have a high degree of analyticity. Our method of expanding this function is based on Bernstein polynomials and is quite different from that of [1].

1. Let D_2 be the complete direct sum $Z_2 \oplus Z_2 \oplus Z_2 \oplus \cdots$, e_n the unit mass at 0 in the nth factor, m_n the uniform $(1/2, 1/2)$ mass in the same group. For a dense sequence $\{a_n\} \subseteq [0, 1]$ we form the infinite product measure

$$\mu = \prod_1^\infty \{a_n e_n + (1 - a_n) m_n\}.$$

Denote by W the set of complex numbers $\{|z| < 1\} - \{-1 < z \leqq 0\}$.

THEOREM. *If f is continuous in $[0, 1]$ and $f \circ \hat{\mu}$ is a Fourier-Stieltjes transform on $\hat{D}_2$, then f can be extended to a function bounded and analytic in W.*

The proof is based on certain measures σ on the N-fold sum $Z_2 \oplus Z_2 \oplus \cdots \oplus Z_2$, in which each element is an N-tuple $(x_1, x_2, \cdots, x_N)$ $(x_i = 0, 1, 1 \leqq i \leqq N)$. Say that σ is *special* if it is invariant with respect to permutations of the coordinates $x_1, \cdots, x_N$. A special measure is a linear combination of the measures σ_j, $0 \leqq j \leqq N$, described as follows: σ_j assigns mass 1 to every element x for which $\sum_{i=1}^N x_k = j$.

For any special measure σ there are defined numbers $b_0, \cdots, b_N$: b_k is the value of $\hat{\sigma}$ on the character

$$x \to (-1)^{\sum_{i=1}^k x_i}.$$

LEMMA. *For a special measure σ, set*

$$B(x) = \sum_0^N b_k \binom{N}{k} x^k (1 - x)^{N-k}.$$