POISSON BOUNDARIES AND ENVELOPES OF DISCRETE GROUPS

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Communicated by G. D. Mostow, January 5, 1967

In [4] we defined the Poisson boundaries for semisimple Lie groups. These spaces play a role in the theory of generalized harmonic functions on the Lie group similar to that played by the boundary of the unit disc in the classical theory of harmonic functions on the unit disc. It is not hard to extend these notions to all separable, locally compact groups, and, in particular, they make sense for countable discrete groups. In this form we shall show that these ideas provide a useful tool for answering certain purely algebraic questions. Namely, we raise the following question. Let G be a connected Lie group, Γ a discrete subgroup for which G/Γ has finite (left-) invariant measure. To what extent is G determined by a knowledge of Γ as an abstract group, and conversely, what is the influence of G on the structure of Γ?

To make this question precise, let us say that G is an *envelope* of Γ if an isomorphic copy Γ' of Γ occurs as a discrete subgroup of G, and $G=D\Gamma'$, where D is a subset of G with finite left-invariant Haar measure. Our question may now be stated in this way. How different can two connected Lie groups G_1 and G_2 be if they both envelop the same countable group Γ?

We shall be discussing a rather restricted version of this question. We suppose that G_1 and G_2 are semisimple and have no compact components, and that G_1 and G_2 envelop the same group Γ. Does it follow that G_1 and G_2 are isomorphic? (Without the hypothesis that G_1 and G_2 have no compact components we could always take $G_2=G_1\times$a compact group.) Our guess is that this is the case. However all we can prove is the following:

THEOREM. *Let H_r, $r=1, 2, 3, \cdots$, denote the hyperbolic group of motions of the r-sphere S^r: H_r consists of the $(r+2)\times(r+2)$ real matrices that leave the form $x_0^2+x_1^2+\cdots+x_r^2-t^2$ invariant. $SL(s, R)$ denotes the group of $s\times s$ real unimodular matrices. If G_1 is one of the groups H_r, $r\geqq 1$ and G_2 is one of the groups $SL(s, R)$, $s\geqq 3$, then G_1 and G_2 cannot simultaneously envelop the same countable group.*

[1] This research was partially supported by the Office of Scientific Research of the Office of Aerospace Research of the USAF under Grant #AF-AFOSR-381-63.