THE VANISHING OF A THETA CONSTANT IS A PECULIAR PHENOMENON

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1. Introduction. The phenomenon in question is the following: (i) if a theta constant θ vanishes at a point t of Torelli space then its gradient (with respect to the coordinates on Torelli space) vanishes there, too; (ii) on the other hand, the locus $\theta=0$ through t is, generically, a hypersurface with tangent plane defined at t, in particular $\theta \not\equiv 0$ on Torelli space.

The reconciliation of (i) and (ii) results from (iii) near t one has $\theta=\Phi^k$, $k>1$ integral, and Φ analytic with nonvanishing gradient at t.

I would speculate that $k=2$, generically, i.e., the locus $\theta=0$ is really the locus $\sqrt{\theta}=0$.

In the next section I shall prove (i). (ii) is in the thesis of Dr. Farkas [**1**], while (iii) is an immediate consequence of (i) and (ii) and some standard algebra in several variables. The speculation on the value 2 for k stems from the appearance of those period relations that are known (Schottky). § 2 is a revision of the remarks in [**3**, pp. 35–37].

2. Definitions and proof of (i). Given a symmetric $g\times g$ complex matrix A with negative definite real part, one can form the Riemann theta function

$$\theta(u, A) = \sum_n \exp(n\cdot An + 2n\cdot u),$$

where n ranges over all integral column g-vectors, u is a column g-vector of complex numbers, and the dot signifies the usual inner product. If, in addition, one is given two column g-vectors ϵ and ϵ' whose entries are 0 or 1, one defines the first order theta function with binary characteristic

$$\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$$

by

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(u, A) = \theta(u+e, A)\exp\left(\frac{\epsilon}{2}\cdot\frac{A\epsilon}{2} + 2\frac{\epsilon\cdot u}{2} + 2\pi i\frac{\epsilon}{2}\cdot\frac{\epsilon'}{2}\right),$$

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