DIFFERENCES OF MEANS

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1. Let $q_1, q_2, \cdots, q_n$ be positive numbers with $\sum_{k=1}^n q_k=1$. For every sequence $(x_1, x_2, \cdots, x_n)$ with all $x_k>0$ and for every real r, consider the *mean of order* r, $M_r(x_1, x_2, \cdots, x_n)$, defined as $(\sum_{k=1}^n q_k x_k^r)^{1/r}$ if $r\neq 0$, and as $\prod_{k=1}^n x_k^{q_k}$ if $r=0$. For given positive $x_1, x_2, \cdots, x_n$, it is known (see, e.g. [3, p. 17], or [11, p. 26]) that $M_r(x_1, x_2, \cdots, x_n)$ is strictly increasing with r (except when $x_1 =x_2 \cdots =x_n$), and consequently if $r<s$, then

$$1 \leqq M_s(x_1, x_2, \cdots, x_n)/M_r(x_1, x_2, \cdots, x_n), \tag{1}$$

$$0 \leqq M_s(x_1, x_2, \cdots, x_n) - M_r(x_1, x_2, \cdots, x_n). \tag{2}$$

2. A natural question to ask is, whether one can give *upper bounds* for the right hand sides of (1) and (2) under, say, the hypothesis that $A \leqq x_j \leqq B, j=1, 2, \cdots, n$, where A and B are constants $(0<A<B)$. Such an upper bound for the ratio in (1) was given by Cargo and Shisha in [4], a paper which served as a motivation and starting point of a considerable amount of further work by various authors.

3. The main purpose of the present note is to give an upper bound for the difference in (2) under the restriction on the x_j stated in §2. As applications, we shall obtain a number of inequalities, including "complements" to the classical inequalities of Cauchy and Hölder. Full proofs omitted here are to be found in [15].

4. In this section, $q_1, q_2, \cdots, q_n$ are fixed (though arbitrary) positive numbers with $\sum_{k=1}^n q_k=1$, and for every sequence $(x_1, x_2, \cdots, x_n)$ with all $x_k>0$ and every real r, $M_r(x_1, x_2, \cdots, x_n)$ is as in §1.

THEOREM 1. *Let r, s, A, B be given reals $(0<A<B, r<s)$, and let I denote the n-dimensional* cube $\{(x_1, x_2, \cdots, x_n)\colon A\leqq x_k\leqq B, k=1, 2, \cdots, n\}$. *Then throughout I,*

$$M_s(x_1, x_2, \cdots, x_n) - M_r(x_1, x_2, \cdots, x_n) \leqq \Delta, \tag{3}$$

where Δ is

$$[\theta B^s + (1-\theta)A^s]^{1/s} - [\theta B^r + (1-\theta)A^r]^{1/r} \qquad \text{if } rs \neq 0, \tag{4}$$

$$[\theta B^s + (1-\theta)A^s]^{1/s} - B^\theta A^{1-\theta} \qquad \text{if } r = 0,$$

and