CONSTRUCTIVE TRANSFINITE NUMBER CLASSES

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1. Introduction. The notion of an ordinal system *restricted productive* with respect to given sets was introduced in [9] and used to define constructive finite number classes. It was shown that both the forms of the sets of notations for the finite number classes and the ordinals obtained are the same as of the sets O, O^O, O^{O^O}, $\cdots$, and the ordinals $\omega_1 < \omega_1^O < \omega_1^{O^O} < \cdots$, respectively. In this article these results are extended to constructive transfinite number classes. We present an ordinal system $(F, |\ |)$ which, in terms of our analogy with the classical ordinals, provides notations for the ordinals less than the first "constructively inaccessible" ordinal. Knowledge of the theory of constructive ordinals suggests that this should lead to a natural class of ordinals of some independent interest. This is born out by the characterization of the ordinals of $(F, |\ |)$ given below. $\boldsymbol{E}_1$ is the type-2 representing functional of the predicate $\lambda\alpha.(\forall\beta)(\exists x)[\alpha(\bar{\beta}(x))=0]$ introduced by Tugué [12] (see also Kleene [4]). Let $\omega_1^{E_1}$ be the smallest ordinal which is not the order type of any well-ordering recursive in $\boldsymbol{E}_1$. Our principal result is that the system $(F, |\ |)$ provides notations for exactly the ordinals less than ω^{E_1}, and the sets of notations for the number classes form an $\boldsymbol{E}_1$-hierarchy.

Kreider-Rogers [5] discussed three systems of notations, each of which regarded internally provides an analogue with the ordinals less than the first inaccessible, but it is not clear that any of these systems gives a natural class of ordinals. It is clear from Theorem 2 below that $(F, |\ |)$ provides notations for at least all of the ordinals of the systems in [5], but the question of equivalence remains open.

Related results are obtained about initial ordinals and hierarchies independent of systems of notations. Proofs will appear elsewhere. Notation used is similar to that of [9].

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