COMPACTLY GENERATED ALGEBRAS OVER DISCRETE FIELDS[1]

BY SETH WARNER

Communicated by R. Arens, October 10, 1966

The structure of locally compact vector spaces over complete division rings topologized by a proper absolute value (or, more generally, over complete division rings of type V) is summarized in the now classical theorem that such spaces are necessarily finite-dimensional and possess the cartesian product topology ([2], [5, Theorems 5 and 7], [1, pp. 27–31]).

Here we present some results on locally compact vector spaces and algebras over infinite discrete fields. Since any topological vector space over a topological division ring K remains a topological vector space if K is retopologized with the discrete topology, some restriction needs to be imposed, and the most natural restriction is straightness: A topological vector space E over a topological division ring K is *straight* if for every nonzero $a \in E$, $\lambda \to \lambda a$ is a homeomorphism from K onto the one-dimensional subspace generated by a. Thus if K is discrete, a straight K-vector space is one all of whose one-dimensional subspaces are discrete.

A category argument establishes the following theorem:

THEOREM 1. *If K is an infinite discrete field, a straight locally compact K-vector space of countable dimension is discrete.*

A field K is *algebraic* if it has prime characteristic and if it is an algebraic extension of its prime subfield.

THEOREM 2. *If K is an infinite discrete division ring, then there exists an indiscrete straight locally compact K-vector space if and only if K is an algebraic field.*

The proof depends on a theorem of Gleason [3, Lemma 1.4.2] concerning the existence of one-parameter subgroups in locally compact groups having no small subgroups and on Jacobson's theorem [4, p. 183] that algebraic division algebras over finite fields are commutative.

A theorem of Pontrjagin [7, p. 153] and Theorem 2 then establish that a straight locally compact vector space over an infinite discrete field is totally disconnected.

[1] Research supported by National Science Foundation grant GP-4473.