THE INNER DERIVATIONS OF A JORDAN ALGEBRA

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A Jordan algebra J is an algebra over a field Φ of characteristic $\neq 2$ with a product $a \cdot b$ satisfying

$$a \cdot b = b \cdot a, \tag{1}$$

$$(a^{\cdot 2} \cdot b) \cdot a = a^{\cdot 2} \cdot (b \cdot a) \tag{2}$$

where $a^{\cdot 2} = a \cdot a$. The following operator identity is easily derived from (1) and the linearized form of (2)

$$[a_R[b_R c_R]] = (a[b_R c_R])_R \qquad \text{for } a, b, c, \in J \tag{3}$$

where x_R denotes right multiplication by x and $[uv] = uv - vu$. Letting $D = [b_R c_R]$, we see that (3) implies $(d \cdot a)D - (dD) \cdot a = d \cdot (aD)$ for $a, d \in J$. In other words, D is a derivation of the Jordan algebra J. Hence every mapping of the form $\sum [b_{iR} c_{iR}]$ is a derivation. We shall call such derivations *inner* derivations and denote the set of all inner derivations of J by $\operatorname{Inder}(J)$. It is easily shown that $\operatorname{Inder}(J)$ is an ideal in the Lie algebra of all derivations of J. We shall show that if the characteristic of Φ is $p \neq 0$, then $\operatorname{Inder}(J)$ is a restricted Lie algebra; that is, $D^p \in \operatorname{Inder}(J)$ if $D \in \operatorname{Inder}(J)$.

If $\mathfrak{A}$ is an associative algebra, we denote by $\mathfrak{A}^+$ the Jordan algebra whose vector space is that of $\mathfrak{A}$ and whose multiplication is $u \cdot v = \frac{1}{2}(uv + vu)$. A Jordan algebra J is *special* if J is a subalgebra of $\mathfrak{A}^+$ for some associative algebra $\mathfrak{A}$. Let $\Phi\{x_1, \cdots, x_n\}$ be the free associative algebra generated by $x_1, \cdots, x_n$ over the field Φ. An element u in $\Phi\{x_1, \cdots, x_n\}$ is called *Jordan* if u is in the subalgebra of $\Phi\{x_1, \cdots, x_n\}^+$ generated by 1 and $x_1, \cdots, x_n$. We can now state the following

LEMMA. *If Φ is of characteristic $p \neq 0$, 2, then there exist Jordan elements $f_i(x, y)$, $i = 1, 2$ in $\Phi\{x, y\}$ such that $[xy]^p = [x, f_1(x, y)] + [y, f_2(x, y)]$.*

PROOF. We introduce the reversal operation in $\Phi\{x, y\}$ which is an involution $a \to a^*$ such that $x^* = x$ and $y^* = y$. We say a is reversible if $a^* = a$. Let $\mathfrak{M}$ be the subspace of $\Phi\{x, y\}$ of all elements of the form $[xa] + [yb]$ where a and b are reversible. Since by Cohn's theorem

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