POSITIVITY OF DUALITY MAPPINGS

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The concept of a duality mapping was introduced by Beurling and Livingston in [1]. A slight generalization of their definition follows.

DEFINITION. *A mapping T from a normed linear space E to its conjugate E^* is called a duality mapping if the following two conditions are satisfied.*

(1) *The direction of $T(x)$ is conjugate to that of x for all x in E, i.e.*

$$\langle T(x), x\rangle = \|T(x)\| \, \|x\|.$$

(2) *There exists an increasing function ϕ from R^+ to R^+ such that*

$$\phi(\|x\| - 0) \leqq \|T(x)\| \leqq \phi(\|x\| + 0),$$

defining $\phi(-0)=0$.

The main theorem of [1] is given below as Theorem 3, with a short, nonconstructive proof. In [2], [3], F. E. Browder has derived the Beurling-Livingstone theorem as a special case of a theorem on monotone operators, i.e. mappings T from E into E^* that satisfy the relation

$$\text{(n)} \qquad \sum_{k\in Z_n} \langle T(x_k), x_k - x_{k-1}\rangle \geqq 0 \text{ for all } (x_1, \cdots, x_n) \in E^n \text{ for } n = 2.$$

Duality mappings are monotone, in fact, they satisfy relation (n) for all natural numbers n. We will call such mappings *positive symmetric.* R. T. Rockafellar has proved in [4] that a mapping T is positive symmetric if and only if it is the subgradient of some convex function G defined on E, i.e. if

$$G(y) \geqq G(x) + \langle T(x), y - x\rangle \quad \text{for all} \quad x, y \in E.$$

The primitive Φ of ϕ, defined by

$$\Phi(t) = \int_{u=0}^{t} \phi(u)\, du \quad \text{for all } t \text{ in } \overline{R^+} = R^+ \cup \{0\}$$

is convex, positive, and increasing. The following theorem thus shows that duality mappings are positive symmetric.

THEOREM 1. *A mapping $T: E \to E^*$ is a duality mapping if and only if for all x in E, $T(x)$ is a subgradient at x of the convex function $\Phi(\|x\|)$, i.e.*

$$\text{(3)} \qquad \Phi(\|y\|) \geqq \Phi(\|x\|) + \langle T(x), y - x\rangle \quad \textit{for all } y \textit{ in } E.$$