PARTITIONS BY REAL ALGEBRAIC VARIETIES, AND APPLICATIONS TO QUESTIONS OF NONLINEAR APPROXIMATION

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In recent investigations concerning Chebyshev approximation one has often considered in place of ordinary polynomials as the allowable approximating functions more general functions. For instance, many authors considered an arbitrary n dimensional linear vector space of continuous functions on some compact set, that is, a family which depends *linearly* on n parameters, and obtained among other things lower bounds for the degree of approximation of certain function classes (see for example Lorentz [1]).

Vitushkin [5] obtained analogous results for more extended families in which the dependence on the parameters is polynomial of higher degree. (He also studied families in which the dependence on the parameters is rational, and even piecewise rational; it turns out that the methods developed for the polynomial case work also in these cases.) To be precise, a subset S of any linear space L is called a "family depending polynomially on n parameters" if the general element of S has the form $P(y_1, \cdots, y_n)$, $y_1, \cdots, y_n$ in the base field of L, where $P(x_1, \cdots, x_n)$ is a polynomial in the indeterminates $x_1, \cdots, x_n$ with elements of L as coefficients. If the polynomial degree of P is d, S is said to depend polynomially of degree d on n parameters. If d equals one, S is of course an n dimensional vector space. It will be assumed below that d is greater than one.

If in addition L is a *normed* linear space with norm $\| \ \|$, one can speak of approximating an arbitrary family F contained in L by a family S depending polynomially of degree d on n parameters. In such a case one says S approximates F to within δ if

$$\sup_{X\in F} \inf_{y_1,\ldots,y_n} \|X - P(y_1, \cdots, y_n)\| = \delta.$$

The above mentioned results of Vitushkin are lower bounds for δ in terms of n and d for certain families F of continuous functions with the supremum norm.

The proofs of Vitushkin's results are quite complicated. The main tools are an estimate of the number of components of a real algebraic variety, due to Oleinik and Petrovsky [2], and results on "variations of sets" as expounded in Vitushkin [4], [5].

The present writer has obtained upper bounds for the partition of n-space by real algebraic surfaces sharper than those directly de-