ON CYCLIC VECTORS OF THE BACKWARD SHIFT[1]

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The forward shift operator U (that is, multiplication by the independent variable) on the space H^2 of the unit circle has been much studied. In particular it is known that a vector f is cyclic (that is, $\{U^n f\}$ $(n \geqq 0)$ spans H^2) if and only if it is an outer function; that every invariant subspace is cyclic; and that two vectors generate the same invariant subspace if and only if they have the same inner factor (see [1, Chapter 5]).

Much less is known about the adjoint operator U^* (the backward shift). The only published result seems to be in [2] where it is shown as a by-product of another investigation that a transcendental entire function is a cyclic vector for U^*. The problem of describing the cyclic vectors for U^* was also posed by D. E. Sarason at the Conference on Analytic Functions, held in Lexington, Kentucky, in May 1965. Both Helson and Sarason have a number of unpublished results on this problem; in particular several of the following theorems were known to them. In this note we indicate some of the results that we have obtained; full details will appear elsewhere.

Let C denote the set of functions in H^2 of the unit circle that are cyclic vectors of U^*; let N denote the set of noncyclic vectors.

THEOREM 1. *N is a dense vector subspace of H^2, and $N + C \subset C$.*

Let Q denote the set of functions on the unit circle that are equal almost everywhere to the quotient of two inner functions.

THEOREM 2. *A function f in H^2 is in N if and only if signum f^2 is in Q.*

COROLLARY. *$(N \cdot N) \cap H^2 \subset N$; $(C \cdot N) \cap H^2 \subset C$; if $f \in C$ and $1/f \in H^2$, then $1/f \in C$.*

Here $N \cdot N$ denotes the set of all products fg with $f, g \in N$.

COROLLARY. *A function f in H^2 is in C if and only if its outer factor is in C.*

COROLLARY. *If $f = \sum a_n z^n \in N$ then $\sum (\operatorname{Re} a_n) z^n \in N$.*

THEOREM 3. *If $\int \log |\operatorname{Re} f| \, d\theta = -\infty$, then $f \in C$.*

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