ON CONVOLUTION AND FOURIER SERIES

BY JACK BRYANT

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In [4, pp. 108–114], Salem found that each function in $L_1(0, 2\pi)$ (or $C[0, 2\pi]$) can be represented as the convolution of a function in L (or C) with an even function in L with convex Fourier coefficients. We announce here a slight generalization of this theorem, and some related results which follow from a study of our methods. Detailed proofs will appear elsewhere [2].

We require the following notation: If f is a function, $(\tau_h f)(x) = f(x+h)$. B will denote a Banach space with norm $\|\cdot\|$. If $f \in L$, $S[f]$ denotes the Fourier series of f, $\{S_n\}$ the partial sums of $S[f]$ and $\{\sigma_n\}$ the $(C, 1)$ means of $S[f]$. $\|\cdot\|_1$ denotes the L_1-norm. If $\{\lambda_n\}$ is a sequence, $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$. We define Q to be the class of functions g with $S[g] = \lambda_0/2 + \sum \lambda_n \cos nx$, where $\Delta^2\lambda_n \geqq 0$ and $\lambda_n \to 0$. Note each function in Q is even, positive, integrable and differentiable on $(0, \pi)$. A will denote an absolute constant, not necessarily the same each time it appears.

THEOREM 1. *Suppose* $S = \sum A_n$ *is summable* $(C, 1)$ *to* f *in a Banach space* B. *Let* ϕ *be a positive increasing function with* $\int_0^\infty 1/\phi(t)dt < \infty$. *Let* $\{\sigma_n\}$ *be the* $(C, 1)$ *means of* S; *if* $\{\lambda_n\}$ *is a sequence such that* $0 < \lambda_n \leqq \phi^{-1}(\|\sigma_n - f\|^{-1})$, $\Delta^2\lambda_n \leqq 0$ *and* $\lambda_n \uparrow \infty$, *then the series* $T = \sum \lambda_n A_n$ *is summable* $(C, 1)$ *in* B.

THEOREM 2. *Let* $B \subset L$ *be a Banach space with* $\|u\|_1 \leqq A\|u\|$ *for each* u *in* B, *and suppose the* $(C, 1)$ *means of* $S[f]$ *are in* B *and* $\|\sigma_n - f\| \to 0$. *Then there exists* $g \in Q$ *and* $h \in B$ *such that* $f = g * h$.

THEOREM 3. *Let* $f \in L$. *Then* $f = g * h$, *where* $g \in Q$ *and* $S[h]$ *and* $S[f]$ *have, except for a set of measure zero, the same points of convergence.*

THEOREM 4. *Suppose* $f \in L$, *and let* $\{\sigma_n\}$ *be the* $(C, 1)$ *means of* $S[f]$. *If* $\sum \|\sigma_k - f\|_1/k < \infty$ *and if* $\|\sigma_k - f\|_1 = o(1/\log k)$, *then* $S[f]$ *converges almost everywhere.*

If we suppose more about B, Theorem 2 can be completed as follows:

THEOREM 5. *Let* $B \subset L$ *satisfy the following conditions*: B *is a Banach space and*

(1) *for each* u *in* B, $\|u\|_1 \leqq A\|u\|$,

(2) *for each* u *in* B *and each* h, $\|\tau_h u\| \leqq A\|u\|$,