ON RINGS OF OPERATORS[1]

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Let $\mathcal{H}$ be a complex Hilbert Space, $B(\mathcal{H})$ the ring of bounded operators on $\mathcal{H}$, E an abelian symmetric subring of $B(\mathcal{H})$ containing the identity which is closed in the weak operator topology, E_1 the commutant of E, and suppose E_1 has a cyclic vector ξ_0 which we normalize so that $|\xi_0| = 1$. Diximier [1] has shown that E (respect. E_1), as a Banach space, is the dual of the Banach space R (respect. R_1) of all linear forms on E (respect. E_1) that are continuous in the ultra-strong topology of E (respect. E_1). In this note we show that every $T \in R$ is also continuous in the weak operator topology of E, from which it follows that a linear functional T on E is continuous in either the weak, ultraweak, strong, or ultrastrong topologies if and only if it is continuous in all four simultaneously. In the process, we obtain an integral representation for such T, which we later use in a theorem on centrally reducible positive functionals on E_1.

We denote the maximal ideal space of E by M, and for $A, B, \cdots \in E$, we denote the corresponding Gel'fand transforms by $a, b, \cdots$. Then $A \rightarrow a$ is an isometric isomorphism from E onto $C(M)$. Consequently, every bounded linear functional on E has the form

$$T(A) = \int_M a(m) d\nu(m), \tag{1}$$

where ν is a complex Borel measure on M uniquely determined by T. Of special interest are functionals of the form $A \rightarrow (A\xi, \xi)$, where ξ is a vector of $\mathcal{H}$. We denote by ν_ξ the measure corresponding to the vector ξ, and by μ the measure ν_{ξ_0}. Then the ν_ξ are all nonnegative, $\|\nu_\xi\| = |\xi|^2$, and it can be seen that $\nu_\xi \ll \mu$ for every ξ. The space M is extremely disconnected, and the measure μ has the special property that $C(M) \cong L_\infty(M, \mu)$ under the natural embedding.

THEOREM 1. *A linear functional T on E is ultrastrongly continuous if, and only if it is weakly continuous, and if and only if there exists a $\phi \in L_1(M, \mu)$ such that*

$$T(A) = \int_M a(m)\phi(m) d\mu(m) \tag{2}$$

for every $A \in E$.

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