MODULAR REPRESENTATION ALGEBRAS[1]

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Let G be a cyclic p-group, K a field of characteristic p, and KG the group algebra of G over K. The *representation ring* $a(KG)$ is generated by symbols $[M]$, one for each isomorphism class $\{M\}$ of finitely generated left KG-modules, with relations

$$[M] + [M'] = [M \oplus M'], \ [M][N] = [M \otimes_K N].$$

The *representation algebra* $A(KG)$ is defined as $C \otimes_Z a(KG)$, where Z is the ring of rational integers, C the complex field. The aim of this note is to give a simple proof of the following theorem of Green [1].

THEOREM. *The representation algebra* $A(KG)$ *is semisimple.*

Since G is a cyclic p-group, the algebra $A(KG)$ is finite dimensional (and commutative), having C-basis $\{v_1, \cdots, v_q\}$, where $q = [G: 1]$, and where $v_r = [V_r]$. Here, V_r denotes the unique indecomposable KG-module of dimension r. We set $A_0 = R \otimes_Z a(KG)$, where R is the real field. Then $A(KG) = C \otimes_R A_0$, and it suffices to prove that A_0 is semisimple, or equivalently, that A_0 has no nonzero elements of square zero.

By the *components* of a module we mean the indecomposable summands in a direct sum decomposition of the module.

LEMMA 1 (ROTH [4], RALLEY [3]). *The number of components of* $V_r \otimes V_s$ *is precisely* $\min(r, s)$.

PROOF. Let H_r be the $r \times r$ matrix with 1's above the main diagonal and zeros elsewhere, let E_r be the $r \times r$ identity matrix, and let λ be an indeterminate over K. Then the number of components of $V_r \otimes V_s$ is the same as the number of invariant factors of $(\lambda E_r + H_r)^s$ different from 1. This easily yields the desired result.

Let us write

$$v_r v_s = \sum_{t=1}^{q} a_{rst} v_t, \qquad 1 \leqq r, s \leqq q.$$

Then the coefficients $\{a_{rst}\}$ are nonnegative integers, and Lemma 1 asserts that

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