ON THE DECOMPOSITION OF INVARIANT SUBSPACES

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1. **Introduction.** The main result of this paper is a decomposition theorem for invariant subspaces of operators on Banach spaces. The theorem is closely related to a decomposition theorem of F. Riesz. We apply the theorem to the study of invariant subspaces of direct sums of operators, producing some new examples of lattices of invariant subspaces of operators on Hilbert space.

2. **The Main Theorem.** Let B be a (complex) Banach space and T a bounded linear operator on B. We use $\sigma(T)$ to denote the spectrum of T. Suppose that $\sigma(T)=\sigma_1\cup\sigma_2$, where σ_1 and σ_2 are disjoint nonempty closed sets. Then, as Riesz has shown, [4, §148], one can choose contours γ_j, $j=1, 2$, with γ_j surrounding σ_j, such that if

$$P_j = -\frac{1}{2\pi i}\int_{\gamma_j}(T-z)^{-1}dz,$$

then P_j is a projection onto an invariant subspace B_j of T. Moreover $B=B_1\vee B_2$, $B_1\cap B_2=\{0\}$, and $\sigma(T|B_j)=\sigma_j$.

We strengthen the hypothesis in Riesz's Theorem and get a stronger conclusion. If S is a compact subset of the complex plane let $\eta(S)$ denote the union of S and all "holes" in S; i.e., $\eta(S)$ is the union of S and all bounded components of the complement of S.

THEOREM 1. *If $\eta(\sigma_1)\cap\eta(\sigma_2)=\varnothing$, then every invariant subspace M of T has a unique decomposition of the form $M=M_1\vee M_2$, where M_j is invariant under T and $M_j\subset B_j$, $j=1, 2$.*

To prove the theorem we require several lemmas.

LEMMA 1. *If $\eta(\sigma_1)\cap\eta(\sigma_2)=\varnothing$, then $\eta(\sigma(T))=\eta(\sigma_1)\cup\eta(\sigma_2)$.*

Lemma 1 follows immediately from a property of the extended complex plane (or 2-sphere) closely related to unicoherence. (For a proof of the relevant property see [6, p. 60].)

Now let M be any nontrivial invariant subspace for T and let R be the restriction of T to M.

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