THE GROTHENDIECK GROUP FOR STABLE HOMOTOPY IS FREE

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Let H_n^m be the set of homotopy types of base-pointed finite complexes of dimension $\leqq m$ and connectivity $\geqq n$. We shall always assume that $2n \geqq m$, in other words, that we are working in the "stable range".

H_n^m is closed under the "wedge" operation ($X \vee Y$ is obtained by identifying the base points in the disjoint union of X and Y). Chang [1] has classified the wedge indecomposables in the case $m \leqq n+3$ and has shown that a unique wedge decomposition theorem holds in H_n^{n+3}, $n \geqq 3$.

PROPOSITION 1. *Unique wedge decomposition fails in H_5^{10}. Indeed $(H_5^{10}, \vee)$ fails to be a cancellation semigroup. The same pathology holds for any H_n^m, $m \geqq n+5$, $2n \geqq m$.*

The easiest example: Let $\nu \in \pi_9(S^6)$ be a map of order 8. Let Cone(ν) be its mapping cone. Then $S^6 \vee \text{Cone}(\nu) \simeq S^6 \vee \text{Cone}(3\nu)$ but Cone(ν) $\not\simeq$ Cone(3ν). (The isomorphism uses only that 3 is prime to the order of ν, the nonisomorphism uses only that 3 is not congruent to ± 1 mod the order of ν. ν could not be of order 2, 3, 4, or 6. Hence a similar example is avoided in the range covered by Chang.)

Let C_n^m be the cancellation semigroup obtained from $(H_n^m, \vee)$ by defining $X \equiv Y$ if there exists Z such that $X \vee Z \simeq Y \vee Z$.

THEOREM 2. *$X \equiv Y$ iff for the bouquet of spheres, B, with the same Betti numbers as X it is the case that $X \vee B \simeq Y \vee B$.*

It follows that the inclusion $H_n^m \to H_n^{m+1}$ remains a monomorphism when we pass to $C_n^m \to C_n^{m+1}$. The suspension functor preserves wedges and hence we obtain a homomorphism from $(H_n^m, \vee)$ to $(H_{n+1}^{m+1}, \vee)$. By Freudenthal's theorem $H_n^m \to H_{n+1}^{m+1}$ is an isomorphism. We obtain a family of monomorphisms $C_n^m \to C_{n'}^{m'}$, $n \leqq n'$, $m \leqq m'$ the direct limit of which we'll call S. Each C_n^m is a sub-semigroup of S and it may be noted that each of the statements below about S and its ambient group specializes nicely to C_n^m and its ambient group.

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