PRODUCING PL HOMEOMORPHISMS BY SURGERY

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In the present note we apply the surgery techniques of [4] to the problem of deforming a homotopy equivalence $f: (M, \partial M) \to (X, \partial X)$ between two p.l. (piecewise linear) manifolds until it is a p.l. homeomorphism. First of all, note that if f can be so deformed, the induced cohomology homomorphism $f^*: H^*(X; Q) \to H^*(M; Q)$ must pull back the rational Pontryagin classes, which are combinatorial invariants; that is, we must have $f^*(p_i(X; Q)) = p_i(M; Q)$ (cf. [1, Chapter XVI]). For the fixed p.l. manifold $(X, \partial X)$ let $\mathrm{PL}^p(X, \partial X)$ denote the "h-cobordism classes" of such homotopy equivalences. Then the problem of deforming f to a p.l. homeomorphism is a special case of the problem of determining $\mathrm{PL}^p(X, \partial X)$. Because if $\mathrm{PL}^p(X, \partial X)$ can be shown to have only one element, then f is h-cobordant to $id: (X, \partial X) \to (X, \partial X)$ and hence homotopic to a p.l. homeomorphism by the p.l. h-cobordism theorem (cf. [5]). The main result (Theorem 1.4) computes a finite upper bound for the number of elements in $\mathrm{PL}^p(X, \partial X)$. Finally, the topological invariance of the rational Pontryagin classes recently demonstrated by S. P. Novikov [3] implies that any topological homeomorphism $f: (M, \partial M) \to (X, \partial X)$ satisfies $f^*(p_i(X; Q)) = p_i(M; Q)$. Thus the Hauptvermutung holds for those p.l. manifolds $(X, \partial X)$ with ord $\mathrm{PL}^p(X, \partial X) = 1$ (cf. Corollary 1.7). Recall that the Hauptvermutung claims that topologically homeomorphic p.l. manifolds are piecewise linearly homeomorphic. Unless otherwise stated we shall work entirely within the p.l. category in what follows.

Let $(X, \partial X)$ be a finite CW pair satisfying Poincaré duality in dimension n. A *piecewise linear structure on the homotopy type of* $(X, \partial X)$, i.e., a *p.l. structure* on $(X, \partial X)$, is a piecewise linear n-manifold $(M, \partial M)$ together with a homotopy equivalence $f: (M, \partial M) \to (X, \partial X)$. Two such p.l. structures $f_i: (M_i, \partial M_i) \to (X, \partial X)$ $(i = 0, 1)$ are *equivalent* provided they are h-cobordant: that is, there is an h-cobordism $(W, \partial_c W)$ between the $(M_i, \partial M_i)$ so that $\partial W = M_0 \cup \partial_c W \cup M_1$ and W (resp. $\partial_c W$) is an h-cobordism between the M_i (resp. ∂M_i); and there is a homotopy equivalence

$$\bar{f}: (W; M_0, \partial_c W, M_1) \to (X \times [0, 1]; X \times 0, \partial X \times [0, 1], X \times 1)$$
$$\text{for which } \bar{f} \mid M_i = f_i \ (i = 0, 1).$$

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