LINEAR FUNCTIONALS ON THE SPACE OF QUASI-CONTINUOUS FUNCTIONS

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Suppose that S is a number interval and J is a nondecreasing sequence of closed and compact number intervals with limit S. Let G denote the space of all quasi-continuous functions from S into the plane. If A is a set then 1_A will denote the characteristic function of A. Let Ω denote the collection of subsets of S to which A belongs only in case 1_AG is contained in G. J has final set in Ω. For each integer n let $|\cdot|_n$ denote the norm for $1_{J(n)}G$ defined by $|f|_n = \text{l.u.b. } |f(x)|$ for all x in $J(n)$. Let $|\cdot|$ denote the function from G to the nonnegative numbers defined by

$$|f| = \sum_{p=1}^{\infty} 2^{-p} |1_{J(p)}f|_p/(1 + |1_{J(p)}f|_p).$$

G is complete in the topology generated by the metric $\rho(f, g) = |f-g|$ and $1_{J(n)}G$ is a closed linear subspace of G for each positive integer n. A linear functional F on G is continuous only in case the restriction of F to $1_{J(n)}G$ is continuous with respect to $|\cdot|_n$ for each positive integer n.

THEOREM. *For each continuous linear functional F on G there is an ordered triple $\{U, V, W\}$ of order additive functions from $S \times S$ to the plane such that if A is in Ω, A is contained in $[a, b]$, and a is not in A, then*

$$F(f) = (L)\int_a^b fU + (I)\int_a^b f(-U + V - W) + (R)\int_a^b fW$$

for each f in 1_AG. Furthermore, if u is an increasing function from $[a, b]$ such that

(1) $U(s-, s) = V(s-, s) = W(s-, s) = 0$, *when s is in $(a, b]$ and* $u(s) = u(s-)$,

and

(2) $U(s, s+) = V(s, s+) = W(s, s+) = 0$, *when s is in $[a, b)$ and* $u(s) = u(s+)$,

and v denotes the function from $[a, b]$ defined by