AXIOMS FOR HOPF INVARIANTS

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1. Since H. Hopf first proved that $\pi_3(S^2) \approx \boldsymbol{Z}$ by showing that a certain invariant was nontrivial, several similar invariants—called generalized Hopf invariants—have been described; indeed, there are no less than 6 different definitions in the literature, and it was not always clear whether theorems about one applied to others. Many definitions were riddled with choices which had to be made, a notable exception being the definition of I. M. James [**5**]. This note gives an axiomatic approach somewhat similar to the axiomatic approach to Steenrod operations. Detailed proofs of the results below, together with some other properties of Hopf invariants, will appear elsewhere.

2. Denote by $\mathcal{C}_+$ the category whose objects are topological spaces with base-point and whose maps are homotopy classes of continuous base-point preserving maps between the spaces. $\mathcal{S}_+$ denotes the subcategory whose objects are countable connected CW-complexes with base-point a vertex; and $\mathcal{F}_+$ denotes the subcategory whose objects are finite connected simplicial complexes with base-point a vertex. In both cases the maps are all the homotopy classes of continuous base-point preserving maps; and we shall suppose this is the case for any subcategory of $\mathcal{C}_+$ considered. If X is a space and A a subspace X/A denotes the identification space obtained by collapsing A to a point and with base-point this point. The circle S^1 with base-point $(1, 0) = e$ lies in $\mathcal{F}_+$. The functors

$$E: \mathcal{C}_+ \to \mathcal{C}_+, \ \vee: \mathcal{C}_+ \times \mathcal{C}_+ \to \mathcal{C}_+, \ \wedge: \mathcal{C}_+ \times \mathcal{C}_+ \to \mathcal{C}_+$$

are defined by (i) $EA = A \times S^1/a_0 \times S^1 \cup A \times e$, (ii) $A \vee B = A \cup B/a_0 \cup b_0$, (iii) $A \wedge B = A \times B/A \times b_0 \cup a_0 \times B$; where a_0, b_0 are the base-points of A, B respectively. Given X, $A \in \mathcal{C}_+$ we shall denote $\mathrm{Hom}_{\mathcal{C}_+}(X, A)$, the homotopy classes of continuous base-point preserving maps of X into A, by $[X, A]$: and if this has the natural structure of a group we shall write $+$ for the operation even when this operation may not be commutative. (Together with induced maps the operation $[\cdot, \cdot]$ defines a functor from $\mathcal{C}_+^* \times \mathcal{C}_+$ to sets.) Moreover, if $X \in \mathcal{C}_+$ we can consider the cohomology theories $H_A^i(X) = [E^i X, A]$, where $i > 0$ and $A \in \mathcal{C}_+$. Then there are pairings between the theories $H_A^*(\cdot)$ and $H_B^*(\cdot)$ to the theory $H_{A \wedge B}^*(\cdot)$ defined as follows. Let $f \in \alpha \in H_A^i(X) = [E^i X, A]$, $g \in \beta \in H_B^j(X) = [E^j X, B]$. Then the homotopy class of