# STABLE COMPLEX MANIFOLDS[1]

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1. T. Van de Ven [3] has recently shown that there exist real 4-dimensional manifolds which admit almost complex structures but admit no complex structures, e.g. $S^1 \times S^3 \# S^1 \times S^3 \# CP(2)$. The purpose of this note is to show that this is an unstable phenomenon.

Let $M^n$ be a $C^\infty$ $n$-dimensional real manifold without boundary and let $\tau_M$ be its tangent bundle. $R^k$ is real Euclidean $k$-space and $C^k$ is complex $k$-space.

DEFINITION 1. $M^n$ admits a *stable complex structure* if $M^n \times R^k$ can be given the structure of a complex analytic manifold for some $k \geqq 0$, $n \equiv k \pmod 2$.

Let $\xi^m$ be an $m$-plane bundle over $M^n$.

DEFINITION 2. A *stable complex structure for* $\xi^m$ is a reduction of the group of $\xi^m \oplus \epsilon^k$ to $U((m+k)/2)$ for some $k \geqq 0$, $m \equiv k \pmod 2$.

DEFINITION 3. A *stable almost complex structure for* $M^n$ is a stable complex structure for $\tau_M$.

PROPOSITION. *$M^n$ admits a stable complex structure if and only if it admits a stable almost complex structure.*

2. It is clear that a stable complex structure carries with it a stable almost complex structure. We show the converse is true.

We can assume $M^n$ is a real analytic manifold. There exists a complex $n$-dimensional manifold $N_c^n$ (of real dimension $2n$) and a real analytic embedding $i: M^n \subset N_C^n$ [4]. Regarding $N$ as a real manifold, it is easy to see from the construction of [4] that $\nu(i)$, the normal bundle of the embedding, is equivalent to $\tau_M$; i.e., $\tau_N | M \approx \tau_M \oplus \tau_M$. Let $U \subset N$ be a tubular neighborhood of $M$ in $N$ and let $r: U \to M$ be the bundle projection ($U$ is identified with the total space of $\nu(i)$). One can construct an open neighborhood, $V$, of $M$ in $U$ $(M \subset V \subset U \subset N)$ which is a domain of holomorphy [2]. Let $r: V \to M$ be the restriction of $r: U \to M$.

Let $\nu^{2n+1}$ be the stable normal bundle to $M^n$. $\tau_M \oplus \nu^{2n+1} \approx \epsilon^{3n+1}$. $\tau_M$ admits a stable complex structure if and only if $\nu$ does. Assume now that there is a bundle $\sigma$ over $M$ with fiber $C^m$ and group $U(m)$

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