CERTAIN HILBERT SPACES OF ENTIRE FUNCTIONS

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1. **Introduction.** The research reported on in the present note was motivated by the following Proposition (F), due to Ernest Fischer ([5], see also [4] for an earlier version; actually Fischer proved a more general result, but the special case suffices as a point of departure for our discussion):

(F) *Let P denote a homogeneous polynomial in $z_1, \cdots, z_k$ with complex coefficients. Then every polynomial in $z_1, \cdots, z_k$ has a unique representation $QP+R$ where*

$$P^*\left(\frac{\partial}{\partial z_k}, \cdots, \frac{\partial}{\partial z_k}\right) R = 0 \tag{1}$$

Here Q, R denote polynomials in $z_1, \cdots, z_k$ and P^* denotes the polynomial whose coefficients are the complex conjugates of those of P.

The proposition (F) underlies various formal schemes for exhibiting a basic set of polynomial solutions of a partial differential equation. (Compare Horváth [8], and for the special case $P=z_1^2+z_2^2+z_3^2$ Hobson [7, Chapter IV]).

Now, if the word "homogeneous" is suppressed, (F) becomes false, since indeed in that Case (1) need not have any nonnull polynomial solution. Nevertheless, if we are willing to abandon the realm of polynomials (F) can be extended. The clue as to how to proceed is provided by Fischer's proof of (F): he defines in the linear manifold of polynomials an inner product with respect to which the operator "multiplication by P" and the differential operator $P^*(\partial/\partial z_1, \cdots, \partial/\partial z_k)$ are adjoint to one another; the required decomposition is then just the orthogonal complement decomposition of the space induced by a pair of adjoint operators. Of course, the polynomials do not form a (complete) Hilbert space with respect to Fischer's inner product. Their completion turns out to be a certain space F_k of entire functions of order two. Within F_k, (F) is now true also in the case that P is not homogeneous, and indeed even for certain entire transcendental functions P, with a suitable interpretation of $P^*(\partial/\partial z_1, \cdots, \partial/\partial z_k)$. The extension of Fischer's result to F_k is far more difficult, however, than the proof of (F) insofar as the problem in F_k is closely intertwined with a series of questions which have no counterpart in the polynomial case. These questions concern adjoints of unbounded