METRIC ENTROPY AND APPROXIMATION[1]

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1. Introduction. The notion of metric entropy (called also ϵ-entropy) has been invented by Kolmogorov [16], [19] in order to classify compact metric sets according to their massivity. The basic definitions are as follows.

Let A be a subset of a metric space X, and let $\epsilon>0$ be given. A family $U_1, \cdots, U_n$ of subsets of X is an *ϵ-covering of A* if the diameter of each U_k does not exceed 2ϵ and if the sets U_k cover A. For a given $\epsilon>0$, the number n depends upon the covering family, but $N_\epsilon(A)=\min n$ is an invariant of the set A. The logarithm

$$H_\epsilon(A) = \log N_\epsilon(A) \tag{1}$$

is the *entropy* of A. (Sometimes this definition is modified by assuming that the sets U_k are balls of radius ϵ.)

Points $y_1, \cdots, y_m$ of A are called *ϵ-distinguishable* if the distance between each two of them exceeds ϵ. The number $M_\epsilon(A)=\max m$ is an invariant of the set A, and

$$C_\epsilon(A) = \log M_\epsilon(A) \tag{2}$$

is called the *capacity of A*. The main general fact about $C_\epsilon(A)$ and $H_\epsilon(A)$ is the simple set of inequalities

$$C_{2\epsilon}(A) \leqq H_\epsilon(A) \leqq C_\epsilon(A). \tag{3}$$

In general, $C_\epsilon(A)$ and $H_\epsilon(A)$ increase rapidly to $+\infty$ as $\epsilon\rightarrow 0$; their asymptotic behavior serves to describe the compact set A.

For the computation of the entropy of concrete sets of functions, Kolmogorov [16], [19], Vituškin [37] and others, have used different special devices. The results obtained were mainly valid for the uniform metric, and for sets A, whose approximation properties by polynomials, or by arbitrary linear combinations of fixed functions were well known. More precisely, the sets A under consideration were sets $A(\Delta, \Phi)$ described below, or at least approximable by such sets.

Let $\Phi=\{\phi_1, \cdots, \phi_n, \cdots\}$ be a *fundamental sequence* of points in

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