A PROPERTY OF THE L_2-NORM OF A CONVOLUTION

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Introduction. It is known that the convolution of two members, f and g, of $L_2(-\infty, +\infty)$ can be a null function without either f or g being a null function. But, if one defines f_ν by setting $f_\nu(x) = e^{i\nu x}f(x)$ for all x, f_ν and g will have a convolution that is not a null function for a suitable choice of ν. There is apparently no information available on how the L_2-norm of the latter convolution depends on ν.

A partial answer to this problem will be provided in the present paper. There will be derived a lower bound on the supremum in ν of the L_2-norm of the convolution of f_ν and g. The lower bound will be expressed in terms of a notion of ϵ-approximate support which is an $L_1(-\infty, +\infty)$ analog of the concept of support of a continuous function on a locally compact space. The inequality will be shown to be sharp in the sense that one can construct an f and a g for which the lower bound is approached arbitrarily closely.

Definitions and notation. Because of the need for uniqueness and because of the nature of the L_1-norm, an appropriate analog for $L_1(-\infty, +\infty)$ of the notion of support is the following.

DEFINITION. *The ϵ-approximate support of a member f of $L_1(-\infty, +\infty)$ is defined to be the closed interval $I_{\epsilon,f}$ such that*

(a) *$I_{\epsilon,f}$ is symmetric about the smallest real number x_0 for which*

$$\int_{-\infty}^{x_0} |f(x)|\, dx = (\tfrac{1}{2})\|f\|_1,$$

(b) $\int I_{\epsilon,f} |f(x)|\, dx = (1-\epsilon)\|f\|_1,$

$\|f\|_1$ being the L_1-norm of f. The existence and uniqueness of x_0 and $I_{\epsilon,f}$ are clear from the absolute continuity of the indefinite integral of $|f|$.

For any Lebesgue-measurable set E the measure of E is denoted by $m(E)$ and the characteristic function is denoted by $\chi(E)$. Given any two measurable functions on the real numbers, f and g, such that for almost all x, $f(y)g(x-y)$ is in $L_1(-\infty, +\infty)$ one denotes by $f * g$ the function for which $(f * g)(x) = \int_{-\infty}^{+\infty} f(y)g(x-y)dy$ a.e. Given any f in $L_1(-\infty, +\infty) \cap L_2(-\infty, +\infty)$ one defines the Fourier transform of f, denoted by $\hat{f}$, by requiring that for all real ω, $\hat{f}(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} \exp(-i\omega x)f(x)dx$. Thus, the definition of $\hat{f}$ for an arbitrary f in $L_2(-\infty, +\infty)$ is determined.