AN ABSTRACT FRAMEWORK FOR THE THEORY OF PROCESS OPTIMIZATION[1]

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Introduction. Ten years ago the development of a maximum principle as a necessary condition for optimality of some control problems began a new era for optimization theory. Since that time different maximum principles have been proposed and proved for a great variety of optimization problems. All these maximum principles and their proofs have a similar structure. The aim of the present paper is to give this unique structure independently of the particular characteristics of any one of these problems.

The present paper is a further addition to the trend started in Gamkrelidze [1] and [2], Halkin [3] and [4], Neustadt [5].

1. **Optimization problem.** We are given a set L, a mapping $f=(f_1, f_2, \cdots, f_k)$ from L into E^k and an integer m with $1 \leqq m \leqq k$. The problem is to find an $\hat{x} \in L$ which maximizes $f_1(\hat{x})$ subject to the constraints $f_i(\hat{x}) \geqq 0$ if $i=2, 3, \cdots, m$ and $f_i(\hat{x})=0$ if $i=m+1, \cdots, k$.

2. **Some assumptions.** The set L is a subset of a linear space X. There is a set $M \subset X$ which is an approximation of L around $\hat{x}$ and a mapping $h=(h_1, \cdots, h_k): X \to E^k$ which is an approximation of f around $\hat{x}$. We shall require that

(i) the set M is convex and $\hat{x} \in M$.

(ii) the functionals h_i are convex for $i=1, \cdots, m$ and linear-plus-a-constant for $i=m+1, \cdots, k$.

(iii) for any set $S=\text{co}\{\hat{x}, x_1, \cdots, x_l\} \subset M$ there is a mapping $\zeta: M \to L$ such that the mappings $f \circ \zeta$ and h are continuous over S (with respect to the usual finite dimensional topology on S) and "tangent at $\hat{x}$ over S" which means that for any $\epsilon > 0$ there is an $\eta \in (0, 1]$ with the property that $|f(\zeta(x)) - h(x)| \leqq \epsilon\delta$ if $\delta \in (0, \eta]$ and $x \in \text{co}\{\hat{x}, \hat{x}+\delta(x_1-\hat{x}), \cdots, \hat{x}+\delta(x_l-\hat{x})\}$.

3. **Maximum principle.** The purpose of the present paper is to prove that there exists real numbers $\lambda_1, \lambda_2, \cdots, \lambda_k$ such that

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