STRUCTURE OF CATEGORIES

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Introduction. This paper sets out to develop a structure theory of categories and carries it, not very far, but far enough for some applications. We need a new definition of *complete* (coinciding with old definitions [2], [8] for well-powered co-well-powered categories). The new definition is needed even to construct images of mappings. With it, we can show that every completion of a small category, and also every primitive category of algebras, is a retract of any category in which it is fully embedded. Such categories are called *injective*; strictly stronger injectiveness properties are rather trivial. By a *completion* of $\mathcal{A}$ is meant a complete category in which $\mathcal{A}$ is fully embedded so that no complete full proper subcategory contains it. The results stated come from the regular completion theory concerning complete extensions of $\mathcal{A}$, regularly represented in Cat($\mathcal{A}^*$, $\mathcal{U}$), and the statements given are simply the main applications of two theorems to the effect that complete categories satisfying certain boundedness conditions are injective. Apparatus is set up, but not developed, for a general completion theory and finer tests for injectiveness.

Precise statements of results cannot well be given before we establish the set-theoretic foundation (§1). The new clause in the definition of completeness requires every intersection of extremal subobjects [8] to be representable, and the dual. Then every completion of a small category $\mathcal{A}$ is well-powered, no intermediate full subcategory is left complete, and the embedding preserves all limits that may exist in $\mathcal{A}$ (and dually). Unfortunately, retraction preserves completeness only in the weaker sense of [8]; and, since not every category has a completion in the same Grothendieck universe, and injectiveness is defined relative to a universe, I can prove that an injective category is complete only in the still weaker sense of Freyd [2]. In any of these senses, up to an equivalence of categories, a left complete full subcategory of a complete category is both left closed in its right closure and right closed in its left closure. Accordingly, one would hope, from Freyd's theorems on existence of adjoints, to retract by a reflector and a coreflector. This question is pursued for

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