THE SOLUTION BY ITERATION OF NONLINEAR FUNCTIONAL EQUATIONS IN BANACH SPACES[1]

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Introduction. Let X be a Banach space, T a (possibly) nonlinear mapping of X into X. We are concerned with the solvability of the equation

$$u - Tu = f \tag{1}$$

for a given element f of X and its relation to the properties of the Picard iterates for the Equation (1), i.e. the sequence $\{x_n\}$ where

$$x_{n+1} = Tx_n + f, \quad x_0 \text{ given.} \tag{2}$$

In a preceding note on the linear case [8], we established the following facts for linear T:

(a) If X is reflexive and T is asymptotically bounded (i.e. $\|T^n\| \leqq M$ for some constant M and all $n \geqq 1$), then the Equation (1) has a solution u for a given f if and only if for any specific x_0, the sequence of Picard iterates $\{x_n\}$ starting with x_0 is bounded in X (see [2]).

(b) For a general Banach space X, if T is asymptotically convergent (i.e. $T^n x$ converges strongly in X for each x in X as $n \to +\infty$), the sequence of Picard iterates $\{x_n\}$ for a given x_0 converges if and only if the equation (1) has a solution.

(c) For a general Banach space X and T asymptotically convergent, if an infinite subsequence of the sequence $\{x_n\}$ converges, then the whole sequence converges to a solution of Equation (1).

Our object in the present note is to give some partial extensions of these results to a general class of nonlinear operators T, and to indicate some interesting examples of the application of these nonlinear results.

THEOREM 1. *Let T be a nonexpansive nonlinear mapping of X into X, (i.e. $\|Tx - Ty\| \leqq \|x - y\|$ for all x and y in X), and suppose that X is uniformly convex. Then the Equation* (1) *has a solution u for a given f in X if and only if for any specific x_0 in X, the sequence of Picard iterates $\{x_n\}$ starting at x_0 is bounded in X.*

PROOF OF THEOREM 1. Let T_f be the mapping of X into X given by $T_f(u) = Tu + f$. Then u is a solution of Equation (1) if and only if

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