THE CONVEX HULL OF A SAMPLE[1]

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Communicated by D. A. Darling, December 16, 1965

1. The convex hull of a random sample may be considered as one possible analogue of the range of a one-dimensional sample. Recent work along this line has dealt with the expected number of vertices, faces, surface area and other quantities connected with the convex hull of n independently and identically distributed random points in the plane and in higher dimensions. See Renyi and Salanke [6] and Efron [1]. Geffroy [2], [3] has shown that if the random points are normally distributed then the convex hull "grows like" an ellipsoid (for a concise statement of these results see Math. Rev. **25** #4559). Some features of the limiting geometric behavior of the convex hull will be described here.

2. Let B be a Banach space. Let $x(1), x(2), \cdots$ be a sequence of independent, identically distributed, Borel measurable, B-valued random variables defined on a probability space $(\Omega, \mathfrak{F}, P)$. Let m be the measure defined on all Borel sets A in B by $m(A) = P(\{\omega : x(1)(\omega) \in A\})$. Let $H(n)$ be the convex hull of $\{x(1), \cdots, x(n)\}$. Let $S(n) = H(n)/\max_{i=1,\cdots,n} \|x(i)\|$, (where $\{0\}/0 = \{0\}$). In other words, $S(n)$ is the random convex hull normalized to have norm one. Let A and C be closed, bounded, convex sets contained in B. We define $d(A, C) = \inf \epsilon$ such that $\epsilon > 0$, $A + \epsilon \supseteq C$ and $C + \epsilon \supseteq A$, where $A + \epsilon = \{y \mid \exists x \in A,\ z \in B \text{ with } 0 \leqq \|z\| < \epsilon \text{ and } y = x + z\}$. We say that C = LIP iff $d(C, S(n)) \to 0$ in probability and that C = LAS iff $d(C, S(n)) \to 0$ almost surely.

It is easy to see that if LAS or LIP exists then it is a compact set. We are thus led to a second, weaker type of limit. Let C be a closed, convex set contained in B. We say that C = WLIP if for all $x \notin C$, $\exists \epsilon > 0$ such that $P(x \in S(n) + \epsilon) \to 0$ and if for all $x \in C$ then $P(x \in S(n) + \epsilon) \to 1$ for each $\epsilon > 0$. We shall say that C = WLAS if for all $x \notin C$, $\exists \epsilon > 0$ such that $x \notin S(n) + \epsilon$ almost surely for all large n, and also if for all $x \in C$ then $x \in S(n) + \epsilon$ almost surely for large n and each $\epsilon > 0$. It is easy to prove that if B is a finite dimensional space then LIP exists iff WLIP exists and the two sets are identical. A

[1] This work was supported by an NSF Fellowship. The research is part of a doctoral thesis written under the direction of Professor John Lamperti at Dartmouth College.