MICROBUNDLES AND THOM CLASSES

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In this note we introduce Thom classes of microbundles. We determine the Thom class of the Whitney sum as the cup product of Thom classes and state two applications; one to Gysin sequences of Whitney sums and one to the Atiyah-Bott-Shapiro duality theorem for Thom spaces (cf. Atiyah [2]). Thus our main result states that for microbundles μ_1, μ_2 over a compact (topological) manifold X, if $\tau(X)\oplus\mu_1\oplus\mu_2$ is J-trivial, then μ_1 and μ_2 have S-dual Thom spaces. Actually our result is more general since it treats the relative case, i.e. with relative Thom spaces. This makes us able to handle manifolds with boundaries (by passing to the double) among other things. Thus proposition (3.2) in Atiyah [2] has an extended version which just appears as another special case of our duality theorem.

The approach given here to the S-duality theorem shows very clearly that S-duality of Thom spaces is simply Alexander-Spanier duality of compact pairs in the base manifold lifted by Thom isomorphisms. Our approach does not make use of imbeddings of manifolds, and we think it is conceptually easier than Atiyah's method, although there are some technical difficulties due to the fact that we work in a more general setting.

Throughout this paper all base spaces of bundles and microbundles are assumed paracompact unless otherwise stated. Manifolds are manifolds without boundary. For notations and concepts see [4].

Generalizations and details will appear elsewhere.

Throughout this paper R denotes a fixed principal ideal domain. By a local system on a space X we understand a local system of R-modules on X, i.e. a (contravariant) functor from the fundamental groupoid of X to the category of R-modules. If

$$\mu: X \xrightarrow{s} E \xrightarrow{p} X$$

is an $\boldsymbol{R}^q$-microbundle with total space E, write $E^0=E-sX$. Then there are local systems $\mathcal{O}=\mathcal{O}(\mu)$, $\mathcal{O}^*=\mathcal{O}^*(\mu)$ on X corresponding under pull-backs and depending only on the equivalence class of μ, such that for $x\in X$, $\mathcal{O}_x=H^q(E|x, E^0|x)$, $\mathcal{O}_x^*=H_q(E|x, E^0|x)$. (Coefficients in homology or cohomology are taken in R if not indicated.)

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