ON THE SPECTRUM AND RESOLVENT OF HOMOGENEOUS ELLIPTIC DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS[1]

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As is well known, in recent years tremendous progress has been made in the study of linear partial differential equations in general and elliptic equations in particular. Powerful existence and regularity theorems have been proved by a number of authors for example Browder [3], [4], Agmon, Douglis, and Nirenberg [1], and Schechter [5], [6], [7], [8]. For general boundary value problems the existence theory has been in the form of alternative theorems and has been limited to relatively compact regions in Euclidean space. To the present author's knowledge there are no general results even for constant coefficient operators in half spaces. It is the purpose of the present paper to make a very small beginning in remedying this lack.

To be precise in the half space $\Omega=\{x\in \boldsymbol{R}^n: x_n>0\}$ let us consider the homogeneous elliptic differential operator with constant coefficients $A=\sum_{|\alpha|=2m} a_\alpha D^\alpha$ and a family of m homogeneous, constant coefficient "boundary operators" B_j, $0\leqq j\leqq m-1$. If $\lambda\in \boldsymbol{C}$, the complex numbers, we ask for necessary and sufficient conditions on λ and the B_j's in order that the map $u\to(A-\lambda)u, B_0u, \cdots, B_{m-1}u$ be a topological isomorphism of $H^{2m}(\Omega)$ onto $H^0(\Omega)\times\prod_{j=0}^{m-1} H^{2m-m_j-1/2}\Gamma$ where $\Gamma=\partial\Omega$ and $m_j<2m$ is the order of B_j. Suppose that the $\{B_j\text{'s}\}$ are such that the map $u\to(Au, B_0u, \cdots, B_{m-1}u)$ has closed graph as a map of $H^0(\Omega)\to H^0(\Omega)\times\prod_{j=0}^{m-1} H^{2m-m_j-1/2}\Gamma$ i.e. such that the usual a priori estimates are satisfied. Then we find that the necessary and sufficient condition in order that the map $u\to(A-\lambda)u, B_0u, \cdots, B_{m-1}u$ be an isomorphism is independent of the particular choice of the $\{B_j\text{'s}\}$ so long as the above mentioned operator has closed graph. As a by-product we find that under these conditions if the operator in H^{2m} defined by A and the null boundary conditions has closed range it is an isomorphism of its domain onto $H^0(\Omega)$.

We will use the following notation: $\boldsymbol{R}$ will denote the real numbers, $\boldsymbol{C}$ the complex numbers. Vectors in $\boldsymbol{R}^n$ will be denoted by $x=(x_1, \cdots, x_n)$ or by $\xi=(\xi_1, \cdots, \xi_n)$ if we refer to the usual dual space. The duality will merely be denoted by $x\xi=\sum_{j=1}^n x_j\xi_j$. If α is an n-tuple of nonnegative integers then $|\alpha|=\sum_{j=1}^n \alpha_j, \xi^\alpha=\xi_1^{\alpha 1}\cdots\xi_n^{\alpha n}$ and $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ where $D_j=(1/i)(\partial/\partial x_j)$. For $u\in\mathcal{S}$ (the space

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