INTERPOLATION SPACES BY COMPLEX METHODS

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1. **Introduction.** We study some methods of constructing interpolation spaces "between" two Banach spaces. The methods extend the complex method introduced by Calderón [1] and Lions [2]. Our results generalize some of those of Calderón [3].

2. **Complex interpolation.** We shall call two Banach spaces X_0, X_1 compatible if they can be continuously embedded in a topological vector space V. We let X_0+X_1 denote the set of those elements $x\in V$ which can be written in the form

$$x = x^{(0)} + x^{(1)}, \tag{2.1}$$

where $x^{(j)}\in X_j$, $j=0, 1$. Set

$$\|x\|_{X_0+X_1} = \inf\{\|x^{(0)}\|_{X_0} + \|x^{(1)}\|_{X_1}\}, \tag{2.2}$$

where the infimum is taken over all pairs $x^{(j)}\in X_j$ satisfying (2.1). One easily checks that (2.2) gives a norm on X_0+X_1 when X_0 and X_1 are compatible. Moreover, when X_0+X_1 is equipped with this norm, it becomes a Banach space (cf. [3]).

Let $\mathcal{A}$ denote the set of complex valued functions of the complex variable $\zeta=\xi+i\eta$ which are continuous on bounded subsets of $\bar{\Omega}$, where Ω is the strip $0<\xi<1$ in the $\zeta=\xi+i\eta$ plane. Let $\mathcal{B}$ be the set of those functions in $\mathcal{A}$ which are holomorphic in Ω and nonvanishing in $\bar{\Omega}$. For $\rho\in\mathcal{A}$ the space $\mathcal{H}(X_0, X_1; \rho)$ will consist of those functions $f(\zeta)$ with values in X_0+X_1 such that (a) $f(\zeta)$ is continuous on bounded subsets of $\bar{\Omega}$, (b) $f(\zeta)$ is holomorphic in Ω, (c) $f(j+i\eta)\in X_j$, $j=0, 1$, η real, and

$$\|f(j + i\eta)\|_{X_j} \leqq \text{const } | \rho(j + i\eta) | . \tag{2.3}$$

Under the norm

$$\|f\|_{\mathcal{H}(X_0,X_1;\rho)} = \max_{j=0,1} \sup_{\eta} | \rho(j + i\eta) |^{-1}\|f(j + i\eta)\|_{X_j}, \tag{2.4}$$

$\mathcal{H}(X_0, X_1; \rho)$ becomes a Banach space.

Let T be a distribution with compact support in Ω (i.e., $T\in\mathcal{E}'(\Omega)$). For $\rho\in\mathcal{A}$ we let $X_{T,\rho}\equiv [X_0, X_1]_{T,\rho}$ denote the set of those $x\in X_0+X_1$ for which there is an $f\in\mathcal{H}(X_0, X_1; \rho)$ satisfying

$$x = T(f). \tag{2.5}$$