REPRESENTATIONS OF COMPLEX SEMISIMPLE LIE GROUPS AND LIE ALGEBRAS[1]

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1. **Notation.** The object of this note is to announce some results on representations of complex semisimple Lie groups and Lie algebras.

$\mathfrak{G}$ is a semisimple Lie algebra over $\boldsymbol{C}$, the field of complex numbers. $\mathfrak{G}$, considered over $\boldsymbol{R}$, the field of real numbers, is denoted by $\mathfrak{G}_0$. $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{G}$, W, the Weyl group of $(\mathfrak{G}, \mathfrak{h})$. We use the standard terminology in the theory of semisimple Lie algebras (Jacobson [**3**] and Harish-Chandra [2(a)], [2(b)], [2(c)]). P_0 is a positive system of roots, fixed once for all and $S_0=\{\alpha_1, \cdots, \alpha_l\}$, the associated fundamental system, $\mathfrak{n}=\sum_{\alpha\in P_0}\mathfrak{G}^{-\alpha}$; $\mathfrak{n}$, considered as a Lie algebra over $\boldsymbol{R}$, is denoted by $\mathfrak{n}_0$. $\mathfrak{h}_0=\sum_\alpha \boldsymbol{R}\cdot H_\alpha$.

Fix a square root $(-1)^{1/2}$ of -1 in $\boldsymbol{C}$. $\mathfrak{k}_0$ is a compact form of $\mathfrak{G}$ containing $(-1)^{1/2}\mathfrak{h}_0$. $\mathfrak{G}_0=\mathfrak{k}_0+\mathfrak{h}_0+\mathfrak{n}_0$ is an Iwasawa decomposition of $\mathfrak{G}_0$ and $G=K\cdot A_+\cdot N$ the corresponding decomposition of G. $c(X\rightarrow X^c)$ is the conjugation of $\mathfrak{G}$ corresponding to the compact form $\mathfrak{k}_0$. Let $\hat{\mathfrak{G}}$ denote the Lie algebra $\mathfrak{G}\times\mathfrak{G}$ over $\boldsymbol{C}$, and let

$$i: X\rightarrow (X^c, X) \qquad (X\in\mathfrak{G}).$$

$(\hat{\mathfrak{G}}, i)$ is a complexification of $\mathfrak{G}_0$. For any $X\in\mathfrak{G}$ let $\overline{X}=(X, X)$, $\overline{\mathfrak{G}}=\{\overline{X}: \overline{\mathfrak{G}}\in\mathfrak{G}\}$. $\hat{\mathfrak{J}}(\mathfrak{J})$ is the universal enveloping algebra of $\hat{\mathfrak{G}}(\mathfrak{G})$ and $\overline{\mathfrak{J}}$ the subalgebra of $\hat{\mathfrak{J}}$ generated by $\overline{\mathfrak{G}}$. For any dominant integral λ, $\mu\in\mathfrak{h}^*$ π_λ denotes the associated irreducible representation of $\mathfrak{G}$ and π_λ^- that of $\overline{\mathfrak{G}}$ (under the isomorphism $X\rightarrow\overline{X}$); $\pi(\lambda, \mu)$ is the irreducible representation $\pi_\lambda\times\pi_\mu$ of $\hat{\mathfrak{G}}$ (Kronecker product).

2. **A theorem on finite dimensional representations.** We have:

THEOREM 1. *Let λ, $\mu\in\mathfrak{h}^*$ be dominant integral, $\nu=\lambda-\mu^*$ and ν^0 the unique dominant integral element in the orbit $w\cdot\nu$. Then the representation $\pi_{\nu^0}^-$ of $\overline{\mathfrak{G}}$ occurs exactly once in the restriction of $\pi(\lambda, \mu)$ to $\overline{\mathfrak{G}}$.*

3. **The homomorphisms h_Q.** For $X\in\mathfrak{G}$ and $a\in\mathfrak{J}$ we write $[X, a]=Xa-aX$. a is said to be of rank 0 if $[H, a]=0$ for all $H\in\mathfrak{h}$. Let $\mathfrak{X}$ be the subalgebra of $\mathfrak{J}$ generated by $\mathfrak{h}$. Suppose Q is any positive system of roots. Then, for any $a\in\mathfrak{J}$ of rank 0, there is a unique $\beta_Q(a)$ in $\mathfrak{X}$ such that $a\equiv\beta_Q(a) \bmod \sum_{\alpha\in Q}\mathfrak{J}\mathfrak{G}^\alpha\cdot a\rightarrow\beta_Q(a)$ is a homo-

[1] The present work was done during 1963–1965 when the authors were at the Indian Statistical Institute, Calcutta.