IDEALS WITH SMALL AUTOMORPHISMS

BY WALTER RUDIN[1]

Communicated November 29, 1965

In [1], Forelli proves the following: If G_1 and G_2 are locally compact Abelian groups, if J is a closed ideal in the group algebra $L^1(G_1)$, and if Ψ is a homomorphism of J into the measure algebra $M(G_2)$ with $\|\Psi\|=1$, then Ψ is induced by an affine map of a coset in Γ_2 into Γ_1. (See [1] for a more detailed statement. For notation and terminology, see [1] or [2]; Γ_i denotes the dual group of G_i; the circle group will be denoted by T.) As Forelli points out in [1], the assumption $\|\Psi\|=1$ cannot be entirely discarded.

Actually, the assumption $\|\Psi\|=1$ *cannot even be replaced by* $\|\Psi\|<1+\epsilon$, *no matter how small* $\epsilon>0$ *is, even if "affine" is replaced by "piecewise affine" in the conclusion, and even if* $G_1=G_2=T$ *and* Ψ *is assumed to be one-to-one.*

Since the integer group Z admits only countably many piecewise affine maps, the preceding statement is a consequence of the theorem below. By way of contrast, it may be of interest to mention that if Ψ is a homomorphism of all of $L^1(G_1)$ into $M(G_2)$ and if $\|\Psi\|>1$, then $\|\Psi\|\geqq\sqrt{5}/2$ [2, p. 88].

THEOREM. *Suppose* $0<\epsilon<1$. *Let* E *be a set of positive integers* λ_k *such that* $\lambda_1=1$ *and*

$$\sum_{k=1}^{\infty}\frac{\lambda_k}{\lambda_{k+1}}<\frac{\epsilon}{6\pi}\cdot \tag{1}$$

Let J *be the set of all* $f\in L^1(T)$ *whose nth Fourier coefficient* $\hat{f}(n)$ *is* 0 *for all* n *not in* E. *Then* J *is a closed ideal in* $L^1(T)$, *with continuum many automorphisms, and every automorphism* A *of* J *(other than the identity) satisfies the inequality*

$$1<\|A\|<1+\epsilon. \tag{2}$$

We shall sketch the proof.

Each A is induced by a permutation α of E. The gaps in E show that no affine map (other than the identity) carries E onto E. Thus $\|A\|>1$ if $A\neq I$.

We write $e(t)$ in place of $e^{2\pi it}$.

Let α be any permutation of Z^+ (the positive integers), let

[1] Supported by NSF Grant GP-3483.