EXTREMAL LENGTH AND REMOVABLE BOUNDARIES OF RIEMANN SURFACES

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1. **Introduction.** Given a Riemann surface R let KD denote the space of harmonic functions u on R with finite Dirichlet norm $\|du\|$ and such that $*du$ is semiexact, i.e., $\int_c * du = 0$ for all dividing cycles c. Then O_{KD} denotes the class of Riemann surfaces R for which every function in KD is constant. Clearly $O_{HD} \subset O_{KD} \subset O_{AD}$ and for planar surfaces $O_{KD} = O_{AD}$. Under various names, this class O_{KD} has been studied by many authors (see, for example, Royden [**4**], Sario [**5**]).

The concept of the extremal length $\lambda(\mathfrak{F})$ of a family $\mathfrak{F}$ of curves on a Riemann surface R can be extended to the case that $\mathfrak{F}$ is a family of curves on the Kerékjártó-Stoilöw compactification $\hat{R}$ of R merely by eliminating the ideal points from each curve. Let α_0, α_1 be compact subsets of R. Define $\hat{\mathfrak{F}}$ to be the family of all arcs on $\hat{R}$ with initial point in α_0 and endpoint in α_1. Define $\mathfrak{F}$ to be the subfamily of $\hat{\mathfrak{F}}$ consisting of all arcs in R. We consider two notions for the extremal distance between α_0 and α_1, viz., define

$$\lambda(\alpha_0, \alpha_1) = \lambda(\mathfrak{F}), \qquad \hat{\lambda}(\alpha_0, \alpha_1) = \lambda(\hat{\mathfrak{F}}).$$

The aim of this note is to announce the following

THEOREM. *A necessary and sufficient condition that* $\lambda(\alpha_0, \alpha_1) = \hat{\lambda}(\alpha_0, \alpha_1)$ *for all compact subsets* α_0, α_1 *of* R *is that* $R \in O_{KD}$.

Our Theorem is reminiscent of the already classical result of Ahlfors-Beurling [**1**]:

A plane point set E is an AD-null set if and only if the removal of E does not change extremal distances.

The relationship between these results will be discussed in §3 below.

2. **Sketch of the proof.** The complete proof will appear in a forthcoming book [**3**]. The main steps in proving the necessity of the extremal distance condition are the following. (i) To construct functions u, $\hat{u}$ on R such that $\lambda(\alpha_0, \alpha_1) = \|du\|^{-2}$ and $\hat{\lambda}(\alpha_0, \alpha_1) = \|d\hat{u}\|^{-2}$, (ii) to show that $R \in O_{KD}$ implies $u = \hat{u}$. (Actually, these steps are applied to each component of $R - \alpha_0 - \alpha_1$, rather than R itself.)

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